F.R. Note No. 131/1955
Research Programme
Objective E3/7 (P)

DEPARTMENT OF SCIENTIFIC AND INDUSTRIAL RESEARCH AND FIRE OFFICES' COMMITTEE
JOINT FIRE RESEARCH ORGANIZATION

HEAT CONDUCTION THROUGH DRY WALLS

by

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Summary

An expression is derived for the temperature rise of the unexposed surface of a wall subjected to fire conditions on one side, the unexposed face of the fire wall losing heat by Newtonian cooling to the atmosphere. The limitations applying to the expression are discussed and it is shown that the expression is valid provided the temperature rise of the cooler surface is less than about half its equilibrium value. Expressions are derived for the fire resistance time of a wall subjected to a B.S. 476 furnace test.

October, 1955.

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Introduction

In this note the temperature rise of the cooler surface of a wall, subjected to fire conditions on one side and to Newtonian cooling to the atmosphere on the other is considered. The solution is obtained analytically, with the aid of a few approximations. The solution would also apply to problems in which the temperature scale had been much reduced, such as in determining the response of the interior temperature of a house resulting from the fall in external temperature conditions with night-fall.

The solution is applicable to other diffusion problems such as, for example, the diffusion of moisture through a wall.

The flow of charge along series resistance, parallel capacity networks is analogous to the flow of heat through walls and thus the solution is again applicable. In this case parallel resistance is analogous to Newtonian cooling in the thermal problem.

In the development of the solution the problem of a lagged wall is first considered. The effect of cooling is then taken into account and a solution derived for the case of walls in which the temperature imposed at the heated surface is not constant with time.

1. The temperature at the far side of a wall heated by a constant temperature.

If one surface of a homogeneous wall is lagged and the other surface is raised to and maintained at a constant temperature θ_1 then the temperature of the lagged surface after time t is given by [1]

$$\theta = \theta_1 \sum_{\eta=0}^{\infty} (-1)^{\eta} \quad 2\,\mathrm{erfc} \sqrt{\frac{rc}{4t(2\eta+1)^2}}$$

where r is the thermal resistance of the wall and c is the thermal capacity of the wall.

This expression is not very tractable but for the purposes of this note can be simplified by only considering temperatures less than $0{\cdot}69\theta_1$ at the unexposed surface. This restriction is justified in solving fire resistance problems since structures would be considered to have failed long before the unexposed surface attained such temperatures.

Thus for $\theta < 0{\cdot}69\ \theta_1$

$$\theta \simeq 2\,\theta_1\ \mathrm{erfc} \sqrt{\frac{rc}{4t}} \qquad (1)$$

Figure 1 is a graph of expression (1) and, as illustrated, can be represented by a straight line with a gradient equal to the gradient at the point of inflection. The expression for this line (see Appendix I) is

$$\theta = \frac{1{\cdot}85\,\theta_1}{rc}\left(t - \frac{rc}{13}\right) \quad \text{--} \qquad (2)$$

If the unexposed face of the wall is unlagged the time temperature curve may again be represented by a straight line making an intercept $\frac{rc}{13}$ with the time axis but having a reduced gradient due to the effect of cooling. (See Appendix II).

The expression for this line is

$$\theta = \frac{\mu\theta_1}{rc}\left(t - \frac{rc}{13}\right) \qquad (3)$$

where μ is less than 1·85 and is given by $\mu = \frac{rc}{\theta_1}\left(\frac{d\theta}{dt}\right)_{max.}$
$\left(\frac{d\theta}{dt}\right)$ Max.
depends on $\frac{r}{r^1}$ and r^1 is the thermal resistance representing cooling.

A graph of μ against $\frac{r}{r^1}$ is given in Figure 2.

Fig. 3 illustrates the solution of a problem by the use of expression (3) compared with a solution obtained by a more laborious method [2].

(2) Validity of the expression for long exposure times.

Where a wall is lagged it can be seen from Fig. 1 that expression (2) can be considered valid at least until $\frac{t}{rc} = 0{\cdot}25$. For unlagged walls where r^1 is large the validity can be considered the same. For larger values of $\frac{r}{r^1}$ the validity is illustrated in Fig. 3. For convenience of calculating validity has been defined as the time at which the rate of temperature rise falls 15% below the maximum rate. (Appendix 3).

In terms of temperature rise the expression is, in general, valid until the temperature rise is about 40% of the equilibrium temperature rise.

(3) Prediction of Fire Resistance

The furnace time-temperature curve to be followed in a fire resistance test is specified in B.S. 476. Where the ambient temperature is 20°C the specification can be met by the following expression:

$$\theta_o = 283 \; t^{1/7} \qquad (4)$$

where t is the exposure time in seconds
and θ_o is the furnace temperature rise in °C

Combining expression (4) and expression (3) by the application of Duhamel's Theorem gives the temperature rise on the far side of a wall subjected to a fire resistance test as

$$\theta = 247 \; \frac{\mu}{rc} \left(t - \frac{rc}{13}\right)^{8/7} \qquad (5)$$

The fire resistance of the wall is taken as the time at which the average temperature of the unexposed surface of the wall rises by 139°C. Inverting expression (5) it is therefore given by

$$t = \frac{rc}{13} + \left(\cdot 561 \; \frac{rc}{\mu}\right)^{\frac{7}{8}} \qquad \text{sec}$$

Although the average temperature rise may be less than 139°C a wall is considered to have failed if the temperature rise at any point on the unexposed surface is greater than 180°C. This may occur for example if the mortar in a brick wall has a greater thermal diffusivity than the brick. The fire resistance is then given by

$$t = \frac{rc}{13} + \left(\cdot 727 \; \frac{rc}{\mu}\right)^{\frac{7}{8}} \qquad \text{sec}$$

where r and c refer to the material with the greatest diffusivity.

Conclusions

If a step function temperature rise, θ_1 is applied to one surface of a dry homogeneous wall with thermal resistance r and thermal capacity c then the temperature rise at the unexposed surface is given by

$$\theta = \frac{\mu\theta_1}{rc}\left(t - \frac{rc}{13}\right)$$

where μ depends on the cooling at the surface.

The fire resistance of a dry homogeneous wall can be predicted from the formula

$$t = \frac{rc}{13} + \left(0.561\,\frac{rc}{\mu}\right)^{\frac{7}{8}} \qquad \text{sec}$$

where failure is by rise in the average temperature of the unexposed face.

Where one portion of the wall has a greater thermal diffusivity than the rest then the fire resistance is given by

$$t = \frac{rc}{13} + \left(0.727\,\frac{rc}{\mu}\right)^{\frac{7}{8}} \qquad \text{sec}$$

where r and c now refer, of course, to the material of greatest thermal diffusivity.

References

(1) Carslaw, H. S. and Jaeger, J. C. Conduction of Heat in Solids. Oxford, 1947, Clarendon Press. para. 116 part I.

(2) Ingersoll, L. R., Zobel, O. J. and Ingersoll A. C. Heat Conduction. New York, 1948. McGraw - Hill Fig. 8-4.

(3) Lawson, D. I. and McGuire, J. H. The Solution of Transient Heat Flow Problems by Analogous Electrical Networks. F.P.E. Note 64/1951.

Appendix I

The Representation of Expression (1) by a Straight Line.

Expression (1) reads

$$\theta = 2\,\theta_1 \,\mathrm{erfc} \sqrt{\frac{rc}{4t}} \qquad (1)$$

and may be represented by a straight line through the point of inflection. The point of intersection will therefore be given by $\frac{d^2\theta}{dt^2} = 0$

$$\frac{d\theta}{dt} = \frac{2}{\sqrt{\pi}} \cdot \frac{\theta_1}{t} \cdot \sqrt{\frac{rc}{4t}} \cdot e^{\frac{-rc}{4t}}$$

$$\frac{d^2\theta}{dt^2} = \frac{d\theta}{dt}\,\frac{1}{t}\,\left(\frac{rc}{4t} - \frac{3}{2}\right)$$

For $\frac{d^2\theta}{dt^2} = 0 \qquad t = \frac{rc}{6}$

and $\theta = 2\mu \,\mathrm{erfc}\,\sqrt{1 \cdot 5}$

The gradient of the curve at this point will be

$$\frac{d\theta}{dt} = \frac{1 \cdot 85}{rc}\,\theta_1$$

Therefore the equation of the required tangent will be

$$\theta = \frac{1 \cdot 85\,\theta_1}{rc}\left(t - \frac{rc}{13}\right) \qquad (2)$$

and the intercept on the t axis will be $\frac{rc}{13}$

Appendix II

The Effect of Cooling

If heat is lost from the surface of a homogeneous wall at a constant rate I then after time t the temperature is lowered by (2)

$$\theta_c = \frac{2I\,r}{\sqrt{rc}} \sqrt{t} \left[\text{ierfc}\,0 + \sum_{\eta=1}^{\infty} 2\,\text{ierfc}\,\eta \sqrt{\frac{rc}{t}} \right]$$

For $t < 0{\cdot}39\,rc$ which corresponds for a lagged wall to $\theta < 0{\cdot}50\,\theta_1$

$$\theta_c = \frac{2I}{\sqrt{\pi}} \frac{r}{\sqrt{rc}} \cdot \sqrt{t} \qquad (6)$$

If the rate of cooling varies and I (t') is the rate at time t' then from (6), by Duhamel's Theorem of heat conduction, at time t

$$\theta_c = \int_0^t \frac{\partial}{\partial t} \left[\frac{2}{\sqrt{\pi}} I(t') \frac{r}{\sqrt{rc}} \sqrt{t - t'} \right] dt'$$

or $$\theta_c = \frac{2}{\sqrt{\pi}} \frac{r}{\sqrt{rc}} \int_0^{\sqrt{t}} I(t - \tau^2)\, d\tau \qquad (7)$$

By Newton's law of cooling the rate of heat loss from a surface is proportional to its temperature

$I = \frac{\theta}{r^1}$ where r^1 is the surface resistance of the wall.

If it is assumed that for a considerable time the rate of temperature rise is constant then

$$\theta = \frac{d\theta}{dt} (t - to) \qquad (8)$$

where $\frac{d\theta}{dt}$ is constant

and $t > to$

Therefore $I = \frac{d\theta}{dt} \frac{(t - to)}{r^1}$

Therefore substituting for I in (7)

$$\theta_c = \frac{2}{\sqrt{\pi}} \frac{r}{\sqrt{rc}} \int_0^{\sqrt{t - to}} \frac{d\theta}{dt} \cdot \frac{t - to - \tau^2}{r^1} d\tau$$

$$\frac{d\theta_c}{dt} = \frac{d\theta}{dt} \frac{2}{\sqrt{\pi}} \frac{r}{r^1} \sqrt{\frac{t - to}{rc}} \qquad (9)$$

If $\frac{d\theta_1}{dt}$ is the rate of temperature rise for a lagged wall and $\frac{d\theta}{dt}$ is the rate in the presence of cooling then

$$\frac{d\theta}{dt} = \frac{d\theta_1}{dt} - \frac{d\theta_c}{dt}$$

Substituting for $\frac{d\theta_c}{dt}$ from (9)

$$\frac{d\theta}{dt} = \frac{\frac{d\theta_1}{dt}}{1 + \frac{2}{\sqrt{\pi}} \frac{r}{r^1} \sqrt{\frac{t - to}{rc}}}$$

Substituting for $\frac{d\theta_1}{dt}$ (Appendix I)

$$\frac{d\theta}{dt} = \frac{\frac{2}{\sqrt{\pi}} \frac{\theta_1}{t} \sqrt{\frac{rc}{4t}}\, e^{-\frac{rc}{4t}}}{1 + \frac{2}{\sqrt{\pi}} \frac{r}{r^1} \sqrt{\frac{t - to}{rc}}}$$

The maximum value of $\frac{d\theta}{dt}$ can be found from (10)

Let $y = \frac{4t}{rc}$ then

$$\frac{d\theta}{dy} = \frac{\frac{2}{\sqrt{\pi}} \theta_1 e^{-\frac{1}{y}}}{y \sqrt{y} \left(1 + \frac{2}{\sqrt{\pi}} \frac{r}{r^1} \sqrt{y - \frac{4to}{rc}}\right)}$$

Therefore $$\frac{d^2\theta}{dy^2} = \frac{d\theta}{dy} \left\{ \frac{2 - 3y}{2y^2} - \frac{1}{2\sqrt{y - \frac{4to}{rc}} \left(\sqrt{\pi} \frac{r^1}{r} + \sqrt{y - \frac{4to}{rc}}\right)} \right\} \quad (11)$$

When $\frac{d^2\theta}{dy^2} = 0$, $\frac{d\theta}{dy}$ is a maximum

When r^1 is large compared with r the intercept on the t axis of the tangent with the maximum slope can be considered the same as for a lagged wall.

i.e. $to = \frac{rc}{13}$ (Appendix I)

Therefore when $\frac{d\theta}{dt}$ is the maximum, from (11)

$$\sqrt{\pi} \frac{r^1}{r} = \frac{y^2}{\sqrt{y - \frac{4}{13}}\,(2 - 3y)} - \sqrt{y - \frac{4}{13}} \qquad (12)$$

From equation (12) the value of t for which $\frac{d\theta}{dt}$ is a maximum can be found for different values of $\frac{r}{r^1}$

The temperature rise can then be found by equation (8)

$$\theta = \frac{\mu \theta_1}{rc} \left(t - \frac{rc}{13}\right)$$

where $\mu = \left(\frac{d\theta}{dt}\right)_{max.} \cdot \frac{rc}{\theta_1}$

and $\left(\frac{d\theta}{dt}\right)_{max.}$ is obtained by substituting for t, to and $\frac{r}{r^1}$ in (10)
Fig. 2. illustrates the variation of μ with $\frac{r}{r^1}$

Appendix III

The Validity of Expression (3) for Long Exposure Times.

The change in rate of temperature rise is proportional to (Appendix II(11))

$$\frac{d^2\theta}{dy^2} = \frac{d\theta}{dy}\left(\frac{1}{y^2} - \frac{3}{2y}\right) - \frac{\frac{d\theta}{dy}}{2\sqrt{y-\frac{4}{13}}\left[\sqrt{y-\frac{4}{13}} + \frac{r^1}{r}\sqrt{\pi}\right]}$$

where $y = \frac{4t}{rc}$

The first term applies to a lagged wall when $r^1 = \infty$ and the second term applies to the effect of cooling.

For long times when the effect of cooling is predominant

$$\frac{d^2\theta}{dy^2} \simeq \frac{-\frac{d\theta}{dy}}{2\sqrt{y-\frac{4}{13}}\left[\sqrt{y-\frac{4}{13}} + \frac{r^1}{r}\sqrt{\pi}\right]}$$

If $\frac{d\theta}{dt}$ falls below $\left(\frac{d\theta}{dt}\right)_{max.}$ by $\delta\left(\frac{d\theta}{dt}\right)$

then $\delta\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}(t - t_m)$ where at t_m, $\frac{d\theta}{dt} = \left(\frac{d\theta}{dt}\right)_{max.}$

therefore the decrease in $\frac{d\theta}{dt}$ is proportional to

$$\delta\left(\frac{d\theta}{dy}\right) = \frac{d^2\theta}{dy^2}(y - y_m) \quad \text{where } y_m = \frac{4t_m}{rc}$$

$$\delta\left(\frac{d\theta}{dy}\right) = \frac{-d\theta}{dy}\,\frac{y - y_m}{2\sqrt{y-\frac{4}{13}}\left[\sqrt{y-\frac{4}{13}} + \frac{r^1}{r}\sqrt{\pi}\right]}$$

Therefore

$$\frac{-\delta\left(\frac{d\theta}{dy}\right)}{\frac{d\theta}{dy}} = \frac{y - y_m}{2\sqrt{y-\frac{4}{13}}\left[\sqrt{y-\frac{4}{13}} + \frac{r^1}{r}\sqrt{\pi}\right]}$$

Let $\frac{\delta\left(\frac{d\theta}{dy}\right)}{\frac{d\theta}{dy}} = \varepsilon$

It is desired to calculate the time at which the temperature rise can no longer be considered as linear. If it is assumed that this happens when the rate of temperature rise falls below the steady rate by approximately 15%

i.e. $\frac{d\theta}{dt} \simeq 0\cdot85\left(\frac{d\theta}{dt}\right)_{max.}$

$$\text{then } \varepsilon = 0\cdot2 = \frac{y_m - y}{2\sqrt{y-\frac{4}{13}}\left[\sqrt{y-\frac{4}{13}} + \frac{r^1}{r}\sqrt{\pi}\right]} \qquad (13)$$

Values of ym for values of $\frac{r^1}{r}$ have been calculated from

$$\sqrt{\pi}\,\frac{r^1}{r} = \frac{y_m^2}{\sqrt{y_m-\frac{4}{13}}\,(2-3y_m)} - \sqrt{y_m-\frac{4}{13}} \qquad \text{(Appendix II (12))}$$

and the corresponding values of y can be found from equation (13).

The variation of deviation time with $\frac{r}{r^1}$ is shown in Fig. 3.

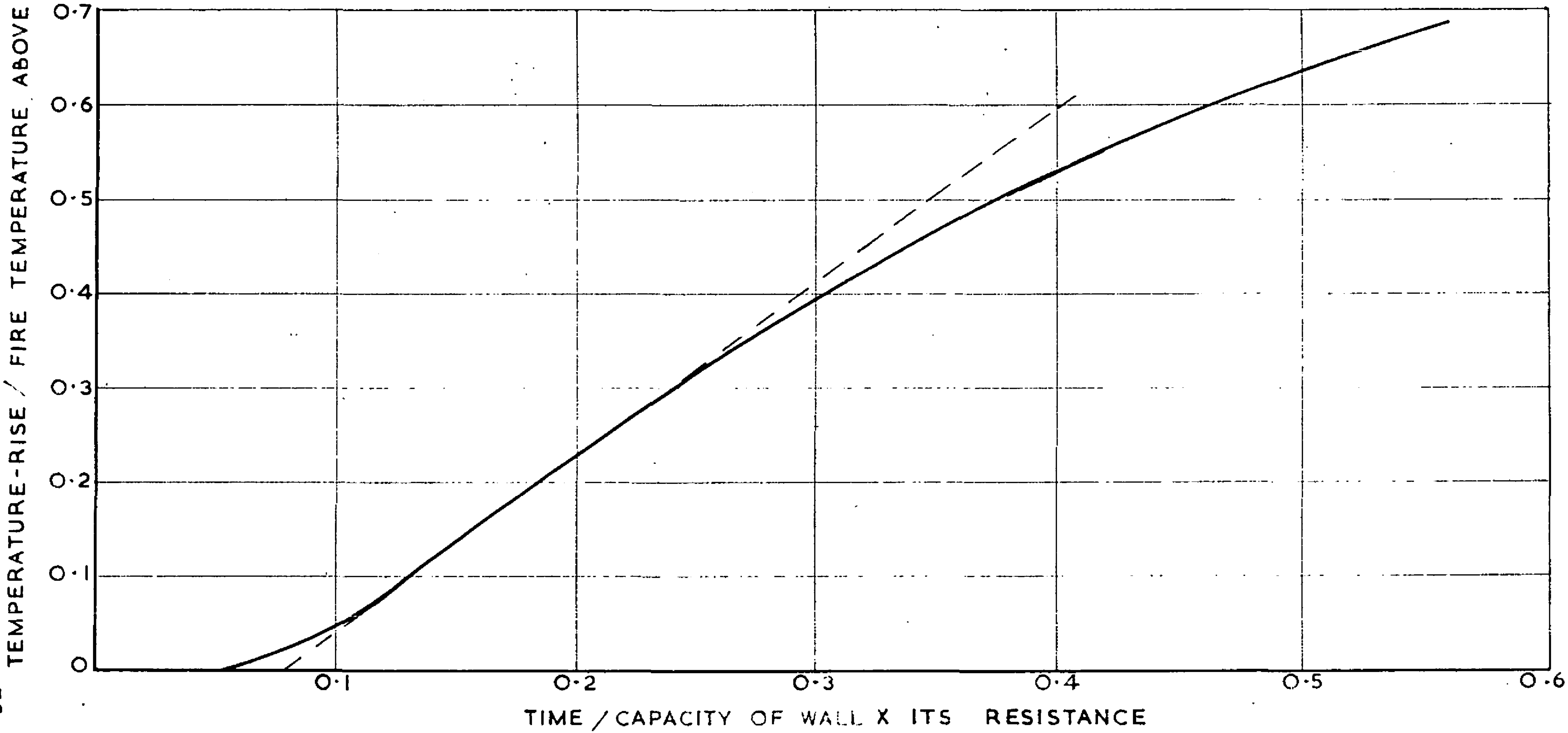


FIG.1. TEMPERATURE RISE AT LAGGED SURFACE OF LAGGED WALL

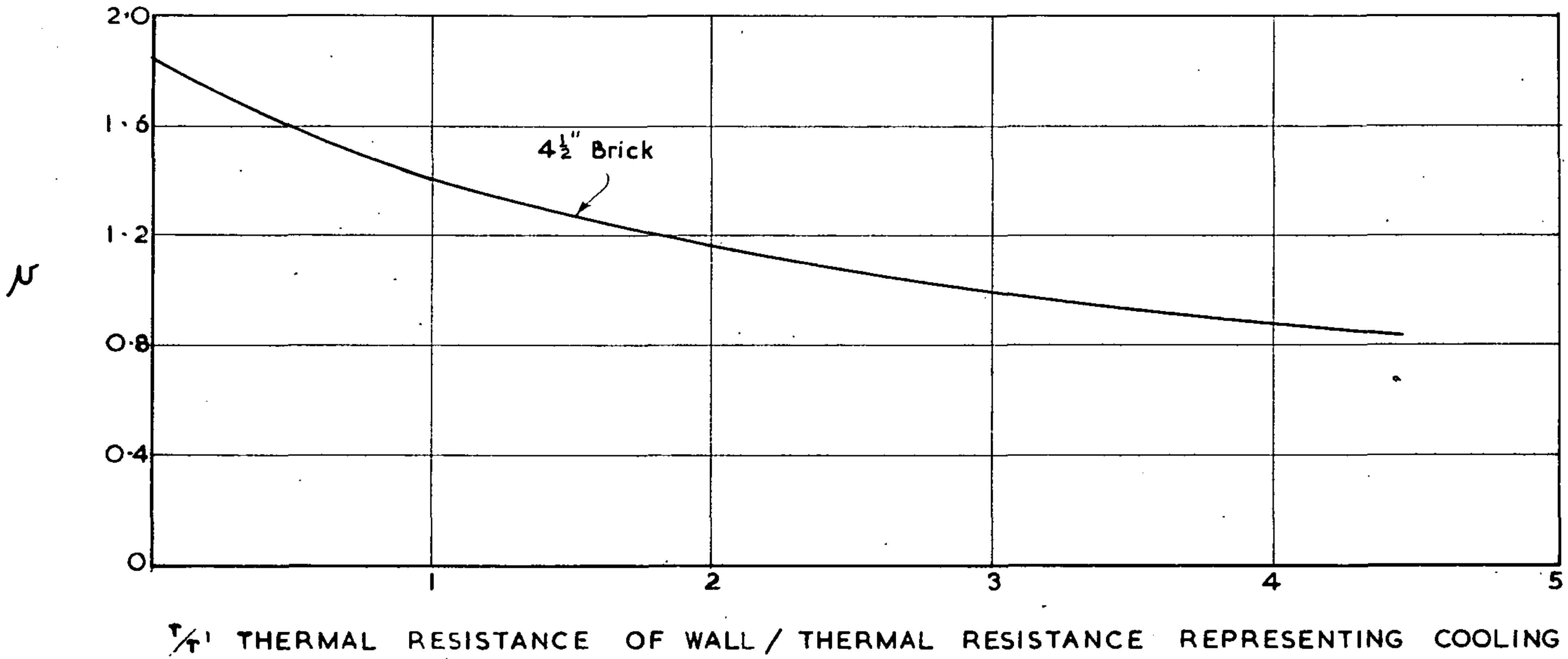
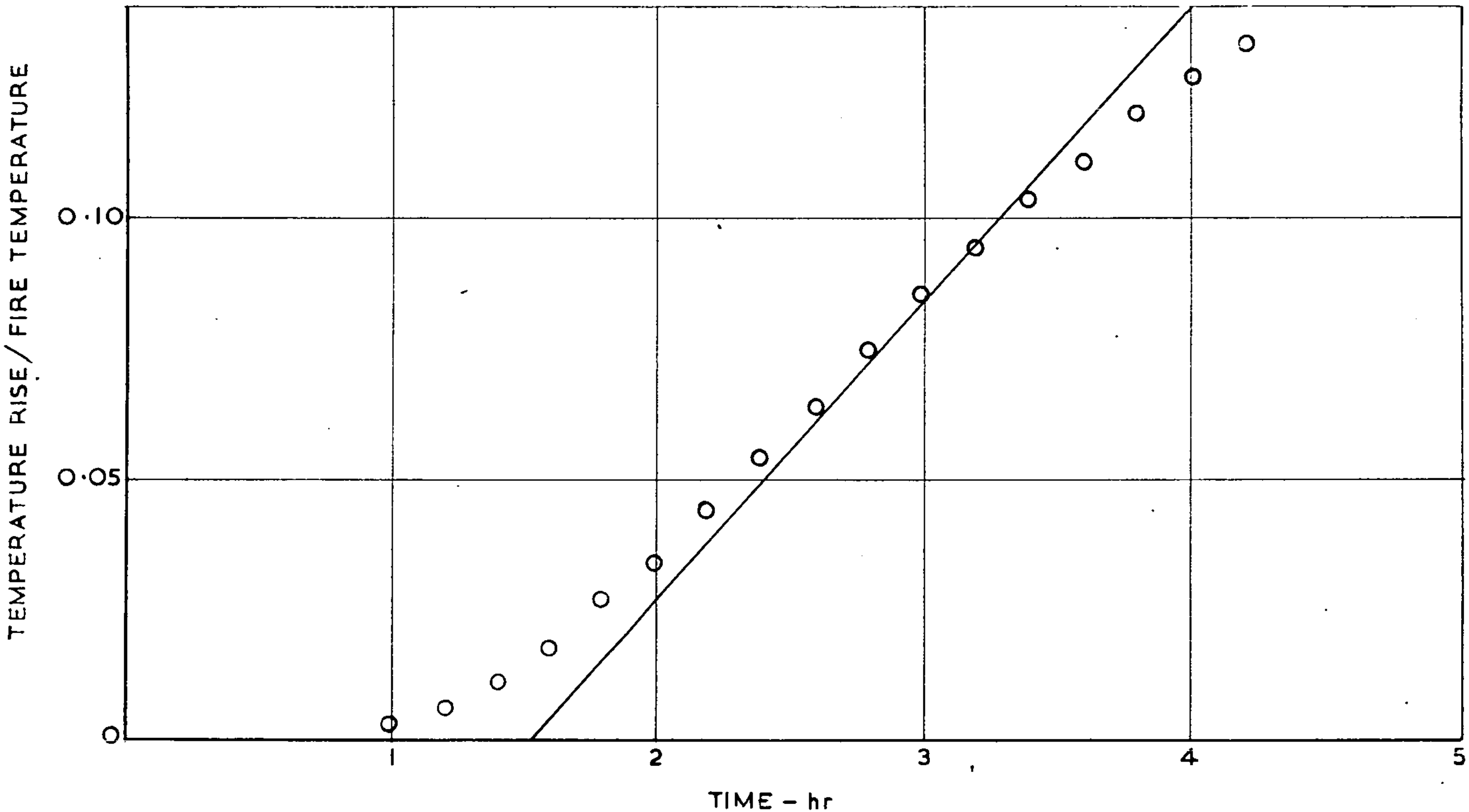


FIG. 2. VARIATION OF μ WITH $\frac{T}{T'}$



O CALCULATED USING AN APPROXIMATE INTEGRAL (2)

—— CALCULATED USING EQUATION (3)

FIG. 3. TEMPERATURE RISE AT THE UNEXPOSED SURFACE OF 8" CONCRETE WALL

$\frac{T}{T'} = 2 \cdot 27$ THERMAL CONDUCTIVITY = $2 \cdot 66 \times 10^{-3}$ C.G.S.

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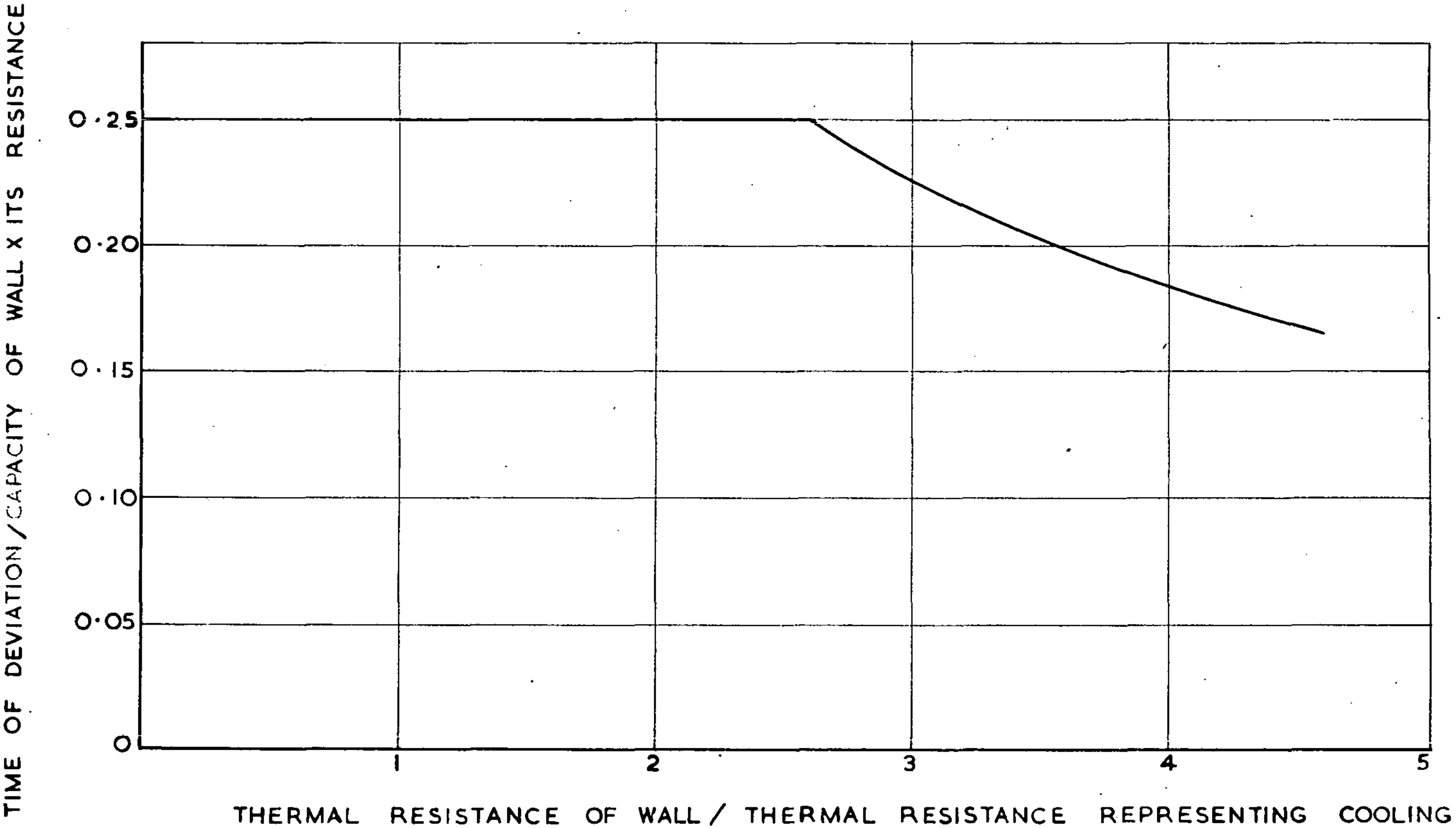

FIG.4. THE DEVIATION FROM STEADY TEMPERATURE RISE