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A NOTE ON THERMAL STRESSES IN BEAMS AND SLABS

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The conventional equations determining thermal stress in beams heated from the top or bottom surface are quoted and the stresses are evaluated for the condition of heating obtained by suddenly raising the temperature of one surface to a steady value. It is shown that even in a reinforced concrete beam failure may result from the attainment of high stresses rather than from failure due to the reaching of a temperature at which the materials lose their strength.

Introduction

The failure of a loaded structure when heated may be due to one or more of several causes. Of these one is the reduction in the strength of the constituent materials when heated above a certain temperature, while another is the setting up of stresses as a result of the tendency of materials to expand when heated. This note outlines the basis for estimating such stresses, the analysis of which is conventional when beams or slabs are heated in one dimension. It is hoped to discuss other problems, such as the two-dimensional heating of part of a surface or of a corner which are relevant to "spalling" in a later report.

Analysis of thermal stress in long beams

The analysis of thermal stresses in a thin plate uniformly heated along its length and breadth is given by Timoshenko and Goodier [1]. The problem has been discussed by other writers notably Lord Rayleigh [2] who evaluated the stresses induced by suddenly changing the temperature on both top and bottom surfaces. Boulton [3] has also discussed the heating of walls and slabs by the application of a constant flux to one surface and equations for the radial heating of solid and hollow cylinders are given by Timoshenko and Goodier [1] and by Case [4]. It may be shown (see Appendix I) that, to a first approximation, the longitudinal and transverse tensile stresses at a height 'y' in a slab of depth ℓ may be represented by

$$f = \frac{E\alpha}{1-\sigma}\left\{\frac{y}{\ell}\left(12V_2 - 6V_1\right) - 6V_2 + 4V_1 - V\right\} \qquad \ldots\ldots (1)$$

where E is the elasticity of the material
σ is Poisson's Ratio
α is the coefficient of linear expansion

$$V_1 = \frac{1}{\ell}\int_0^{\ell} V\,dy \qquad \ldots\ldots (2)$$

$$V_2 = \frac{1}{\ell^2}\int_0^{\ell} V\,y\,dy \qquad \ldots\ldots (3)$$

and V is the temperature at a height 'y'.

This stress satisfies the condition that the resultant longitudinal and transverse thrusts and bending moments are zero. If the beam is narrow so that there are only longitudinal forces to be considered, the stress is given by an equation similar to the above except for the absence of the term '$1 - \sigma$'.

These equations apply to an unrestrained and unloaded beam, but the effect of constraints and loading can be dealt with by application of the principle of superposition. If the effect of longitudinal thrust on the deflection of the beam is neglected it may be shown that the load-deflection equation for a constrained and loaded beam is the same as that for a similarly constrained beam with an initial curvature given by

$$\frac{1}{R} = \frac{6\alpha}{l}\left(2V_2 - V_1\right) \qquad \ldots\ldots (4)$$

It is sufficient therefore to evaluate the temperature-time distribution and the integrals V_1 and V_2 to determine the transient stress distribution.

If the rise in temperature of the unheated face is small the temperatures may be regarded as those in a semi-infinite solid so that for a beam heated on its upper face by raising it to a temperature V_S

$$V = V_S \operatorname{erfc} \frac{l-y}{2\sqrt{kt}} \qquad \ldots\ldots (5)$$

where 't' is the time
'k' is the thermal diffusivity

The integrals V_1 and V_2 can now be obtained as

$$V_1 = \frac{2}{\sqrt{\pi}}\sqrt{\frac{kt}{l^2}}\, V_S \qquad \ldots\ldots (6)$$

$$V_2 = V_1 - \frac{kt}{l^2} V_S \qquad \ldots\ldots (7)$$

If these values of V_1 and V_2 are substituted in equation (1) we can plot $\frac{(1-\sigma)p}{E\alpha}$ as a function of kt/l^2 for various values of $\epsilon\ (= y/l)$. This is shown in Figure (1). The distribution of stress at two values of kt/l^2 other than zero is also shown in Figure (2). For example if "t" is taken as 30 minutes, the diffusivity of concrete as 0·0022 cm^2/sec and l as 20 cm, kt/l^2 is 10^{-2} approximately. The maximum +ve value of $\frac{(1-\sigma)p}{\alpha E}$ after $\frac{1}{2}$ hour is thus seen in Figure (1) to have reached 0·21 at $\epsilon = 0.7$. (3/10 of the beam thickness below the heated surface). For concrete

$\alpha = 10^{-5}$ per °C approximately
$E = 4 \times 10^6$ lb.sq.in. approximately.

so that in an unstrained concrete beam exposed to a surface temperature of, say, 500°C tensile stresses of $0{\cdot}21 \times 4 \times 10^6 \times 10^{-5} \times 500$ or 4,200 lb.sq.in. will have been developed within $\frac{1}{2}$ hour.

This stress is several times the ultimate strength of concrete in tension and unless the compressive loading and constraining stresses are of this order to compensate for the induced thermal stress, there will be a failure of the beam. This analysis only applies to a homogeneous beam but the relevant expressions for a reinforced beam may be obtained on similar lines as follows.

Composite beam

The beam will be considered as shown in Figure 3.

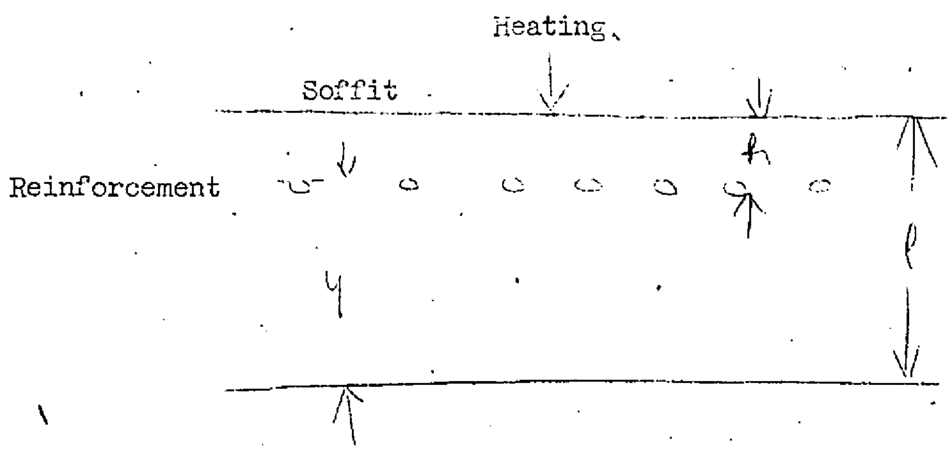

Figure 3

The area of reinforcement is equivalent to a thickness "s" for the whole width of beam. The temperature of the reinforcement will be considered uniform over its area and will be denoted by V_r. (Suffix "c" refers to concrete and suffix "r" to the reinforcement). As before we write the tensile strain in the concrete as linear in y. (See equation (10) Appendix (1)).

i.e.

$$p_c = E_c\left(a + by/\ell - \alpha_c V_c\right) \qquad \ldots\ldots (8)$$

where 'a' and 'b' are functions of temperature only. The equilibrium equations for the horizontal load and the bending moment in the beam are given respectively by

$$\int_0^{\ell} p_c\,dy + s\,p_r = 0 \qquad \ldots\ldots (9)$$

$$\int_0^{\ell} p_c\,y\,dy + s\,p_r(\ell - h) = 0 \qquad \ldots\ldots (10)$$

The stress in the reinforcement must provide a net strain equal to that in the concrete at the level of the reinforcement (1 - h). Thus we have

$$p_r = E_r\left(a + b\left(1 - \frac{h}{e}\right) - \alpha_r V_r\right) \quad \text{......(11)}$$

In solving equations 8-11 two approximations have been made for convenience. The first of these assumes that

$$\alpha_r V_r \ll a + b\left(1 - \frac{h}{e}\right) \quad \text{......(12)}$$

This is equivalent to neglecting the temperature strain as compared with the gross strain in the steel and is discussed below.

If the temperature drop in the beam occurs near the heated surface

$$V_1 \doteq V_2 \quad \text{......(13)}$$

This is seen from equations (6) and (7) to be justified for $\frac{ht}{e^2} \ll 1$.

The solutions of equations (8-11) for the two constants a and b give

$$\frac{a}{\alpha_r V_1} = \frac{\phi \frac{b}{\alpha_r V_1}\left(1 - h/e\right) - 1 + \frac{b}{2\alpha_r V_1}}{1 + \phi} \quad \text{......(14)}$$

$$\frac{b}{\alpha_r V_1} = \frac{\frac{1}{2} + \phi\, h/e}{\frac{1}{12} + \phi\left(\frac{1}{3} - \frac{h}{e}\left(1 - \frac{h}{e}\right)\right)} \quad \text{......(15)}$$

where $$\phi = \frac{E_r}{E_c}\frac{S}{e} \quad \text{......(16)}$$

If a rectangular beam with all its reinforcement at one level is designed so that the maximum compressive stress in the concrete due to its working load is p_c and the maximum tensile stress in the reinforcement due to its working load is p_r, it can be shown (see reference (4) p. 166-7) that ϕ is given by

$$\phi = \frac{\gamma^2}{2(1 - \gamma)} \quad \text{......(17)}$$

where $$\gamma = \frac{1}{1 + \frac{p_r}{p_c}\frac{E_c}{E_r}} \quad \text{......(18)}$$

If the beam is designed so that p_r/p_c is the ratio of the maximum permissible stresses i.e. there is an optimum design of beam, ϕ is approximately 0·1.

If l is 20 cm and h is 5 cm then from equations (14) and (15) we have

$$\frac{b}{\alpha_c' V_1} = 5\cdot 37$$

and

$$\frac{a}{\alpha_c V_1} = -1\cdot 89$$

From equation (6) and the above values for a and b the inequality (12) is satisfied as long as

$$\frac{\alpha_r}{\alpha_c}\,\mathrm{erfc}\,\frac{h}{2l}\sqrt{\frac{l^2}{kt}} \ll \frac{2}{\sqrt{\pi}}\left(5\cdot 37\left(1-\frac{h}{l}\right)-2\cdot 07\right)\sqrt{\frac{kt}{l^2}}$$

......(19)

Since the coefficient of expansion for steel (α_r) and concrete (α_c) are about the same

$$\frac{\alpha_r}{\alpha_c} \doteqdot 1. \qquad ......(20)$$

we may write (19) for $\frac{h}{l} = \frac{1}{4}$ as

$$\mathrm{erfc}\,\frac{1}{8x} \ll 2\cdot 1\, x \qquad ......(21)$$

where x is $\sqrt{\frac{kt}{l^2}}$

This inequality is satisfied for $x < 0\cdot 06$, i.e. for 0.4 x 10^{-2} (say). For times equivalent to these values it will be noted that . For concrete with = 20 cm this corresponds to times less than about 10 minutes. The tensile stresses in the concrete at the unheated surface will then be given by

$$p_c = \frac{2E\alpha_c V_s}{\sqrt{\pi}}\left[5\cdot 37\frac{y}{l} - 1\cdot 89\right]\sqrt{\frac{kt}{l^2}} \qquad .(22)$$

The compressive stresses at the unheated surface due to a heated face of 500° will therefore be

$$p_{c\,max} = \frac{2}{\sqrt{\pi}} \times 40 \times 500 \times 1\cdot 89 \times \sqrt{\frac{kt}{l^2}} \qquad ...(23)$$

$$= 4\cdot 27 \times 10^4 \sqrt{\frac{kt}{l^2}} \quad \text{lb.sq.in.}$$

i.e. in 10 minutes the induced compressive stress at the unheated face will reach 2,600 lb.sq.in. This is clearly of a magnitude that might cause failure in an already loaded beam. The tensile stress in the steel reinforcement is given by

$$p_r \doteq E_r (0{\cdot}75\ b + a)$$

$$= 2{\cdot}1\ E_r \alpha_c V_s \sqrt{\frac{kt}{\rho^2}} \quad \ldots\ldots(24)$$

After 10 minutes, $p_r \doteq 2 \times 10^4$ lb.sq.in.

i.e. 9 tons/sq.in. approximately.

This again would cause failure in a loaded beam. In practice a beam in a fire is heated by having its temperature raised gradually so that the stresses will not be so great as estimated here. They would, however, be expected to be of a high order and sufficiently large to make the criterion of failure as the attainment of a fixed temperature by the steel an over-simplification. It would be worth while computing V_{λ} V_1 & V_2 exactly for the standard time temperature curve for fire resistance so that the stresses may be evaluated more exactly and for longer times if necessary.

References

1. Timoshenko and Goodier. "Theory of Elasticity". McGraw-Hill, 1951. p. 399.
2. Lord Rayleigh. Phil.Mag.S.6 (1901) 1 page 169.
3. Boulton. "Temperature stresses in walls and slabs". Phil.Mag. S.6 (1933) 16 p. 145.
4. Case. "Strength of materials". Arnold.

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APPENDIX I

The differential equation (1) for a two dimensional thermal stress system in which there are no stresses p_z, q_{xz}, q_{yz} is

$$\nabla^4 \chi + E\alpha \nabla^2 V = 0 \qquad \text{......(1)}$$

where $\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2}$

and χ is the stress function.

In particular the tensile stresses in the x and y direction are

$$p_x = \frac{d^2\chi}{dy^2} \qquad \text{......(2)}$$

and

$$p_y = \frac{d^2\chi}{dx^2} \qquad \text{......(3)}$$

and the shear stress $q_{xy} = -\frac{d^2\chi}{dx\,dy}$

Equation (1) may be written as the pair of equations

$$\nabla^2\chi + E\alpha V = \psi \qquad \text{......(4)}$$

$$\nabla^2\psi = 0 \qquad \text{......(5)}$$

From (4) we have

$$p_x + p_y + E\alpha V = \psi \qquad \text{......(6)}$$

We shall not attempt to solve (4) and (5) exactly for a finite beam but an approximate solution may be derived for a long, unrestrained weightless beam at a distance from its ends.

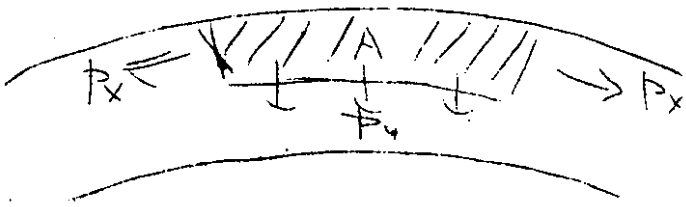

Figure 4

From considerations of symmetry we have zero shear on the three faces of element A of the beam (Figure (4)) and we have no variations in stress or strain with x.

Hence, since p_x, p_y and V are functions of y only

we have from (6) $$\frac{d\psi}{dx} = 0 \qquad \ldots\ldots(7)$$

and hence from (5) and (7) $$\psi = a + b\,y/\ell \qquad \ldots\ldots(8)$$

where 'a' and 'b' are constants.

Since strain in both x and y directions is a function of y only it follows that layers of different y may be considered concentric. If R is the radius of curvature of the bottom of the beam and $R \gg \ell$ we can to a first approximation use cartesian co-ordinates with the additional equation of equilibrium of the element in Fig. (4).

$$p_y + \frac{1}{R}\int_y^{\ell} p_x\, dy = 0 \qquad \ldots\ldots(9)$$

Clearly, from equation (9) p_y is small compared with p_x if $\frac{\ell}{R} \ll 1$.
From equations (6), (8) and (9) we have for $\frac{\ell}{R} \ll 1$

$$p_x = a + b\,y/\ell - E\alpha V \qquad \ldots\ldots(10)$$

we also have from the equilibrium of the beams

$$\int_0^{\ell} p_x\, dy = 0 \qquad \ldots\ldots(11)$$

$$\int_0^{\ell} p_x \cdot y \cdot dy = 0 \qquad \ldots\ldots(12)$$

From equations (10), (11) and (12) we then obtain

$$p_x = E\alpha\left\{\frac{y}{\ell}\left(12V_2 - 6V_1\right) - 6V_2 + 4V_1 - V\right\} \qquad \ldots\ldots(13)$$

where $$V_1 = \frac{1}{\ell}\int_0^{\ell} V\,dy \qquad \ldots\ldots(14)$$

and $$V_2 = \frac{1}{\ell^2}\int_0^{\ell} V y\,dy \qquad \ldots\ldots(15)$$

For a system of plane strain as in a wide slab E is replaced by $E/1-\sigma$ in the above equations.

It is now possible to evaluate the maximum tensile value of p_y. This occurs at the value of "y" nearer to the heated surface where p_x is zero. (The points X in Fig. (2).). We can identify E/R with the coefficient of "y" in the expression for p_x and thereby find from equation (9) by a graphical integration of Fig. (2) the maximum tensile value of p_y. Thus we have

$$p_y \text{ max.} = 0{\cdot}08\, E\alpha^2 V_s^2 \text{ for } \frac{kt}{\ell^2} = 4{\cdot}5 \times 10^{-3}$$

and $$\text{max.} = 0{\cdot}11\, E\alpha^2 V_s^2 \text{ for } \frac{kt}{\ell^2} = 9 \times 10^{-3}.$$

The ratio of these stresses to the maximum tensile values of p_x are $0{\cdot}42\,\alpha V_s$ and $0{\cdot}51\,\alpha V_s$. For the values of α and V_s used above it is clear that the horizontal tensile stresses are considerably greater than those normal to the surface.

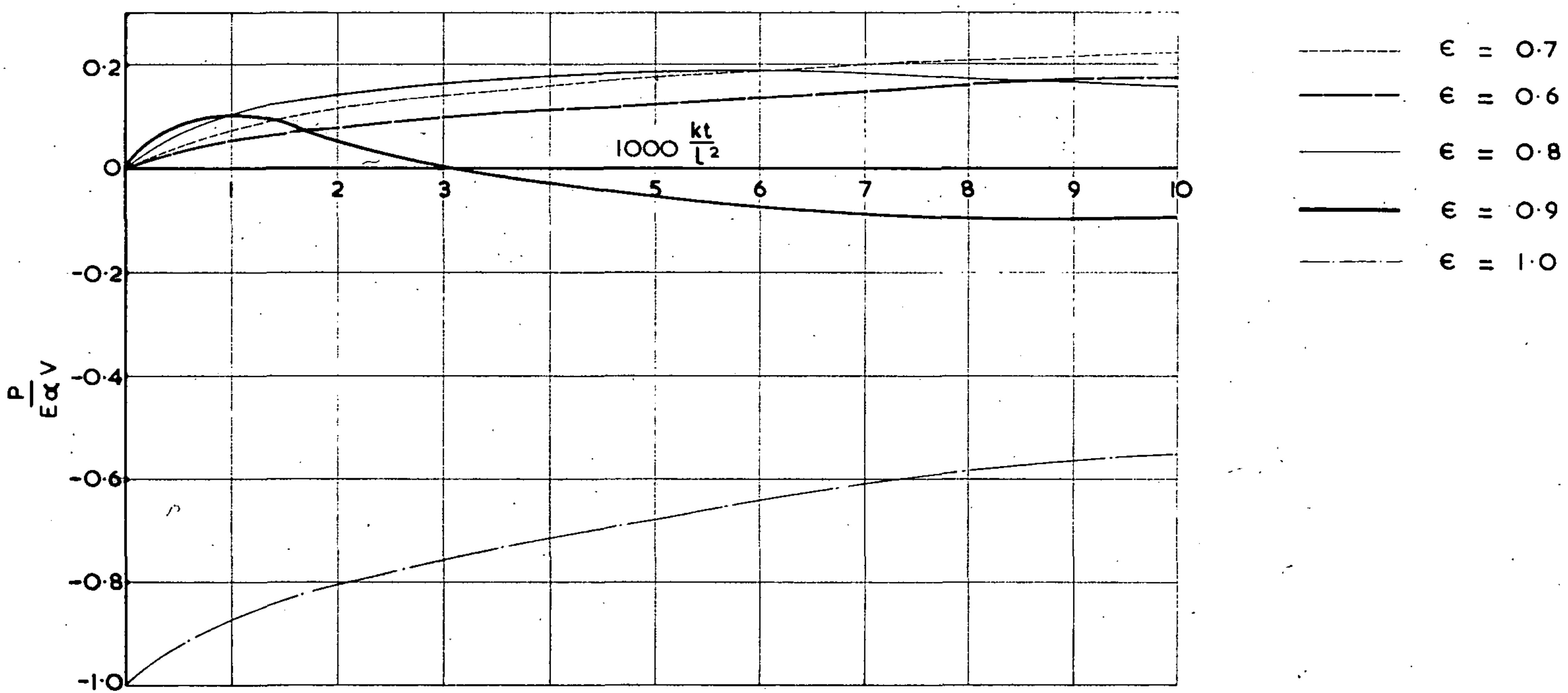

FIG. I. LONGITUDINAL STRESSES IN A LONG NARROW UNRESTRAINED BEAM HEATED BY SUDDENLY RAISING ONE SURFACE TO A STEADY TEMPERATURE.

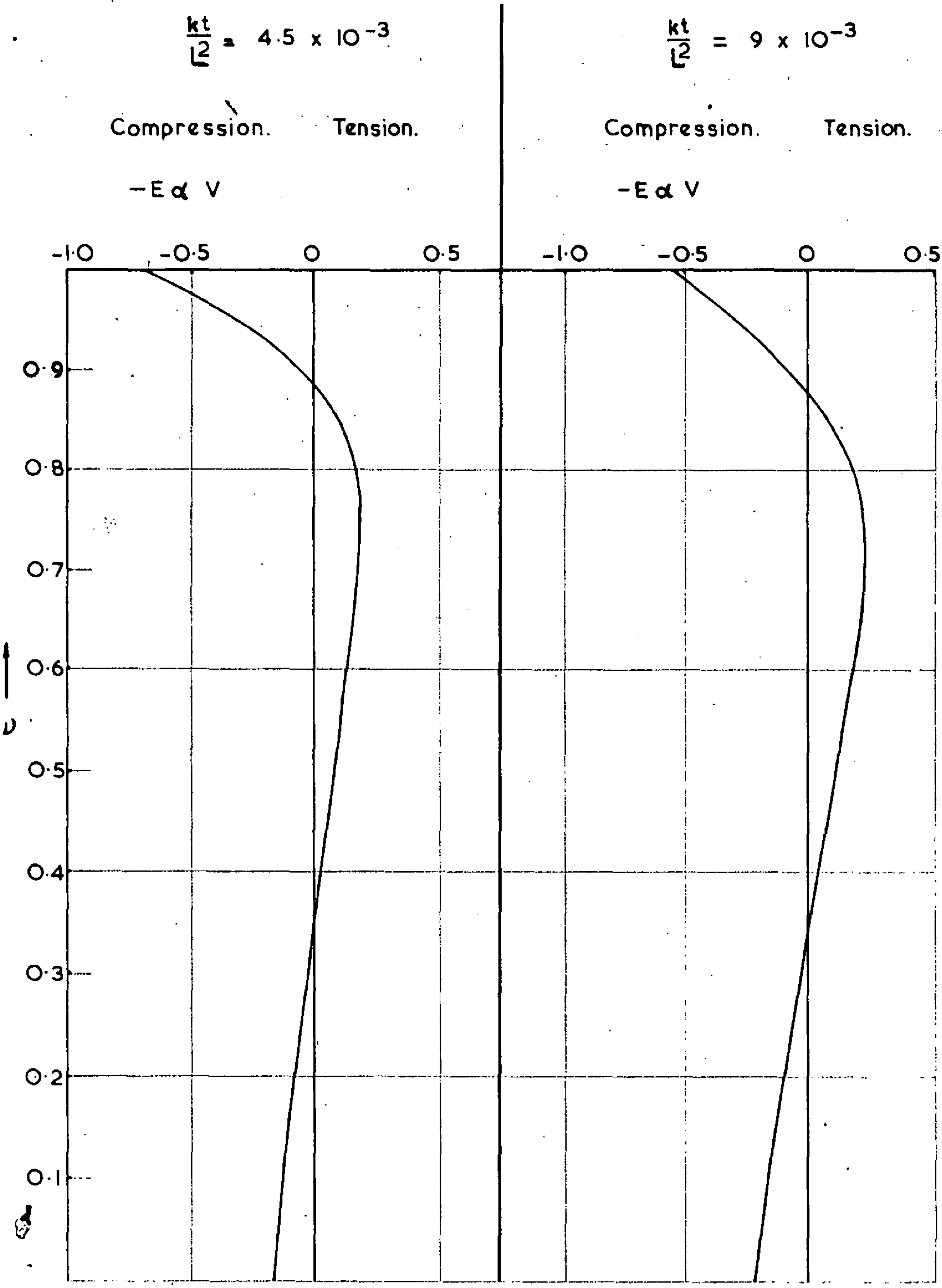

FIG. 2: DISTRIBUTION OF LONGITUDINAL STRESSES THROUGHOUT DEPTH OF A NARROW BEAM.

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