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F.R. Note ~~No. 163/1955~~
Research Programme.
Objective B/2/1(P)

SOME PRACTICAL LIMITATIONS TO THE USE OF SMALL IRRADIATED AREAS ON SOLIDS FOR THE STUDY OF THERMAL DAMAGE

by

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1. Introduction

In order to obtain sources of high intensity radiation it is necessary to use optical systems which can only provide small areas of uniform irradiation. It is important to know what effect this limited area of irradiation has on the temperature rise of the surface since this controls the thermal damage suffered.

In this paper, two forms of the problem of the conduction heat flow from a circular heating source on a semi-infinite solid are discussed. In the first, analogous to the classical problem of the electrified disk, the temperature or potential V is uniform over a circular area and zero in the remainder of the surface plane. In the second and more useful problem, the flux is uniform over the circular area and zero elsewhere in the plane. More complex boundary conditions present difficulties in analysis and, even for the two above, a complete analytical result is only possible in certain conditions. Although only circular sources are discussed, there are some results for square sources given by Jaeger, but there is little difference between them for equal areas [1].

2. Theoretical analysis

The differential equation for the temperature "V" is

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{k}\frac{\partial V}{\partial t} \qquad (1)$$

the boundary conditions for $z = o$ being

either (a) $V = 0, \quad r > R \quad t > o$

$\qquad = V_o \quad r < R \quad t > o$

or (b) $-K\left(\frac{\partial V}{\partial z}\right)_0 = 0 \quad r > R \quad t > o$

$\qquad = Q \quad r < R \quad t > o$

with the initial condition $V = 0$ at $t = o$ for $Z > 0$

The notation is conventional throughout.

Condition (a) is Weber's classical problem of the field due to an electrified disk.

2.1 Condition (a)

In the steady state, this leads to the solution (2)

$$V = \frac{2V_0}{\pi} \cot^{-1} \zeta \tag{2}$$

where $$Z = \mu \zeta$$

and $$r = (1-\mu^2)^{1/2} (1+\zeta^2)^{1/2}$$

Alternatively the solution may be written (2)

$$V = \frac{2V_0}{p} \int_0^\infty \frac{e^{-pz/R}}{p} J_0(pr/R) \sin p \, dp \tag{3}$$

where J_0 is the Bessel function of the first kind of order zero. The heat flux into the solid can be obtained from (3) and is given by

$$-K\left(\frac{\partial V}{\partial x}\right)_0 = \frac{2KV_0}{\pi R} \int_0^\infty J_0(pr/R) \sin p \, dp \tag{4}$$

$$= 0 \qquad r > R$$

$$= \frac{2KV_0}{\pi\sqrt{R^2 - r^2}} \qquad r < R \tag{5}$$

The mean heat flow is thus

$$Q_m = \frac{2K}{R^2} \int_0^R \left(\frac{\partial V}{\partial z}\right)_0 r \, dr = \frac{4KV_0}{\pi R} \tag{6}$$

which is twice the flux at the centre.

2.2 Condition (b)

The solutions for condition (b) are most easily obtained by considering point sources of flux Q per unit area per second distributed over the circular area ($r <$ R) over the surface Z = 0, (3)

$$V = \frac{Q}{4\rho c}\int_0^t d\lambda \int_0^{2\pi} d\theta \int_0^R \frac{r_1 e^{-\frac{r_1^2 + r^2 - 2rr_1\cos\theta}{4k\lambda}}}{(\pi k\lambda)^{3/2}} dr_1 \quad (7)$$

For the steady state, V can be found analytically on the surface, but for a finite time "t" V can only be found analytically at the centre of the source (r = 0, Z = 0).

From equation (7) the steady state surface temperature is

$$V_s = \frac{Q}{4\rho c}\int_0^R r_1 dr_1 \int_0^{2\pi} d\theta \int_0^{\infty} \frac{e^{-\frac{r^2 + r_1^2 - 2rr_1\cos\theta}{4k\lambda}}}{(\pi k\lambda)^{3/2}} d\lambda \quad (8)$$

For $r < R$

$$V_s = \frac{2}{\pi}\frac{QR}{K}E\left(\frac{r}{R}\right) \quad (9)$$

and for $r >$

$$V_s = \frac{2QR}{\pi K}\left[F\left(\frac{R}{r}\right) - E\left(\frac{R}{r}\right)\right] \quad (10)$$

where $$F(x) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - x^2\sin^2\theta}} \quad (11)$$

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*These solutions in equations (9) and (10) are obtained by integral substitutions, given by G. N. Watson "Theory of Bessel Functions" C.U.P. (1944).

and
$$E(x) = \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2\theta}\, d\theta \tag{12}$$

F and E are the complete elliptic integrals of the first and second kinds respectively.

The form of this temperature distribution is shown in Figure (1).

At the centre of the disk, we have from equation (9)

$$V = \frac{qR}{K} \tag{13}$$

The mean temperature over the area $r < R$ is, from equation (9)

$$V_m = \frac{4}{\pi} \frac{q}{KR} \int_0^R r E\left(\frac{r}{R}\right) dr \tag{14}$$

$$= \frac{8}{3\pi} \frac{qR}{K} \tag{15}$$

2.3 Transient solutions

For $r = Z = 0$ in equation (7), the solution, which is given by Blok,[4] is

$$V_0 = \frac{qR}{K} \left\{ \frac{2}{\sqrt{\pi}} \left(\frac{kt}{R^2}\right)^{1/2} \left[1 - e^{-R^2/4kt}\right] + \operatorname{erfc} \frac{R}{2\sqrt{kt}} \right\} \tag{16}$$

For $\frac{kt}{R^2} \ll 1$, this approximates to

$$V_0 \doteqdot \frac{2Q}{K} \left(\frac{kt}{\pi}\right)^{1/2} \tag{17}$$

This is the solution for a constant flux entering a semi-infinite solid over the whole surface[5]. V_0 is shown as a function of kt/R^2 in figure (2).

The Inversion Integral is taken round the contour shown in Figure (4)

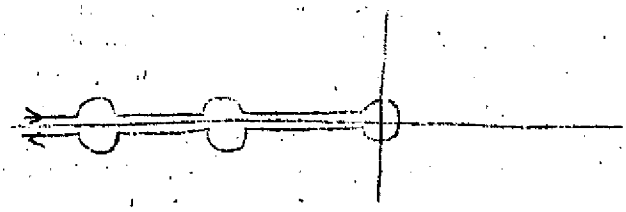

Figure (4). Contour of integration

with poles at $p = -\dfrac{4\pi^2 n^2 k}{R^2}$ $(n = 0, 1, 2, 3 \text{ etc.})$

The solution is finally obtained after some reduction as

$$Q(t) = Q(\infty)\left[1 + \frac{1}{\sqrt{\tau}} + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \tau}\right] \tag{23}$$

where $$Q_\infty = \frac{KV}{R} \tag{24}$$

and $$\tau = \frac{4\pi k t}{R^2} \tag{25}$$

For small values of τ equation (23) reduces to

$$Q \doteq \frac{2Q_\infty}{\sqrt{\tau}} \tag{26}$$

This is identical with the result for the heating of the whole surface of a semi-infinite solid [6]. The values of $Q(t)$ given by equations (23) and (26) are shown in Figure (5).

5. Cooling at the surface

5.1 Steady state

The boundary condition to be satisfied at the surface $Z = 0$ is now

$$HV - K\left(\frac{\partial V}{\partial z}\right)_0 = Q \qquad r < R$$
$$= 0 \qquad r > R$$

where H is the cooling coefficient.

An important consequence of this result is, that if experiments using small heat sources as in work on ignition and burns are to be valid for large areas, the times involved in the experiments must not exceed a certain amount; otherwise heat loss by conduction at right angles to the direction of heating will become relatively large. From Figure (2) or equations (16) and (17) the relative value of the temperature of the equivalent plane and the disk for various values of kt/R^2 can be found. This is shown in Figure (3). The relevance of this is discussed below.

3. Uniform transient heat flow

Another problem of interest is to find the uniform transient heat flow over a circular area applied so that the temperature at the centre is raised to a constant value

In a similar manner to equation (7), the temperature at the centre of the area

$$V = \frac{1}{4\rho c}\int_0^{2\pi} d\theta \int_0^R r_1\, dr_1 \int_0^t \frac{Q(\lambda)\, e^{-r_1^2/4k(t-\lambda)}}{\pi k(t-\lambda)^{3/2}}\, d\lambda \tag{18}$$

$$= \frac{1}{\rho c}\int_0^t Q(\lambda)\, \frac{\left(1 - e^{-\frac{R^2}{4k(t-\lambda)}}\right)}{\left(\pi k(t-\lambda)\right)^{1/2}}\, d\lambda \tag{19}$$

Using the Laplace transform with V equal to V_1 for all " " and denoting the transform of Q by $\bar{Q}$

$$\frac{V_1}{p} = \frac{\bar{Q}}{\rho c(\pi k)^{1/2}}\int_0^\infty \frac{e^{-pt}}{t^{1/2}}\left(1 - e^{-\frac{R^2}{4kt}}\right) dt \tag{20}$$

$$= \frac{\bar{Q}}{\rho c(kp)^{1/2}}\left[1 - e^{-R\sqrt{p/k}}\right] \tag{21}$$

Hence $$\bar{Q} = \frac{V_1\, \rho c\, k^{1/2}}{p^{1/2}\left(1 - e^{-R\sqrt{p/k}}\right)} \tag{22}$$

The solution to equation (1) for the steady state condition satisfying this condition is

$$V = \frac{QR}{K}\int_0^\infty \frac{e^{-\alpha z} J_0(\alpha r) J_1(\alpha R)\, d\alpha}{h+\alpha} \tag{27}$$

where $h = H/K$

We can evaluate this integral at the central point $r = 0$, $z = 0$.

Thus

$$V_c = \frac{QR}{K}\int_0^\infty \frac{J_1(\alpha R)}{h+\alpha}\cdot d\alpha \tag{28}$$

$$= \frac{QR}{K}\left[1 + hR + \frac{\pi}{2}\left(Y_1(hR) - {}^{*}H(hR)\right)\right] \tag{29}$$

where Y_1 is Weber's Bessel function of the second kind of the first order

and H_1 is Struve's function of the first order.

$\frac{KV}{QR}$ is shown for various values of hR in Figure (6).

Since the cooling term occurs as the product it is evident that small disks are less affected by surface cooling than large ones. In the limit as hR becomes infinite we obtain the surface temperature

$$V_c = \frac{QR}{K}\,\frac{1}{hR}$$

$$= \frac{Q}{H} \tag{30}$$

which is the value for the heating of a plane.

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*This solution is obtained from "Theory of Bessel Functions". G. N. Watson. C.U.P. (1944).

5.2 Cooling - transient conditions

From Figure (6) it is seen that for values of hR greater than 6, the steady state disk and plane temperatures are effectively equal. Since the temperatures are also effectively equal for small values of kt/R^2 it is thus possible to say that provided hR is greater than about 6, disks may be considered equivalent to planes throughout their heating.

Since the surface temperatures in the transient state are less than in the steady state, the effect of surface cooling would be expected to be less in the transient state.

Thus at a time when the disk and plane temperatures diverge noticeably the temperature of the plane is reduced by cooling by a greater amount than is the temperature of the disk, and this results in a decrease in the divergence so that it may be said that the presence of cooling prolongs the time for which the disk and plane are effectively equal. For values of hR less than 6 the time for which disk and plane are effectively equal is calculated for the condition of zero surface cooling. This then gives a conservative estimate in practice when there is cooling.

6. Application of results

The theoretical results given above are generally useful in experimental studies involving small irradiated areas. In particular Blok's result for the transient temperature of the centre of a small disk ($hR << 6$) gives a measure of the extent to which a disk and a plane source are comparable. If experiments involve times of irradiation T_i then the source area must exceed the value given by

$$a_{min} = \frac{\pi k T_i}{N(\epsilon)} \qquad (31)$$

where $N(\epsilon)$ is the value of kt/R^2 giving the tolerable difference ϵ. The required value of N is found from Figure (3). Alternatively, the largest irradiation time T_i permitted with an area a is

$$T_{i\,max} = \frac{aN}{\pi k}$$

It is now possible to discuss various experimental assemblies in the light of this discussion. Table (1) lists four such radiation sources.

TABLE 1

Parameters of various experimental assemblies

Source of radiation and reference	Radius of source R cm	K cal cm^{-1} $^{\circ}C^{-1}$ sec^{-1}	H cal cm^{-2} $^{\circ}C^{-1}$ sec^{-1}	$h = H/K$ cm^{-1}	hR	Maximum time at which ignition or burn occurs (secs) and corresponding "error"	
(11) Lamp	0.56	OAK 4×10^{-4}	2.5×10^{-3}	6	3.5	25	13%
		FIBREBOARD 2×10^{-4}	2.5×10^{-3}	12	7		
(12) Searchlight (FRO)	0.18	OAK 4×10^{-4}	3.6×10^{-3}	9	1.62	2.5	13%
		FIBREBOARD 2×10^{-4}	3.6×10^{-3}	18	3.24	2.0	13%
(9) (10) Searchlight (Rochester)	0.56	SKIN (7) 1×10^{-3}	3×10^{-4}	0.15	0.8	100 50	12% 4%
(B) Searchlight (USNRDL)	$\alpha = 1.78$ $\beta = 1.5$ $\alpha\beta = 1.2$	OAK 4×10^{-4}	3.6×10^{-3}	9 9 9	16 13.5 10.8		
	$\alpha = 1.78$ $\beta = 1.5$ $\alpha\beta = 1.2$	FIBREBOARD 2×10^{-4}	3.6×10^{-3}	18 18 18	32 27 21.6		

The equivalent radius "R" produced by each source has been calculated and is quoted in Column (2). For the typical materials tested with each apparatus (viz. oak, fibreboard and skin) (7) approximate values of thermal conductivity "K" and cooling coefficient "H" are quoted (Columns (3) and (4) respectively). The value of H is calculated on the following assumptions. The temperature rise developed is assumed as the ignition temperature in the case of woods. This may be of the order of 500° to 1000°C. For the work on burns (7) the temperature rise is small (say 25°C) and the heat loss is assumed to be entirely convective whereas with the higher temperatures, developed in the ignition of materials, radiation is predominant. In the case of radiation cooling it is only possible to assume a maximum value of H given by the black body heat loss at the ignition temperature.

It is seen from Column (6) that the value of hR for the experiments at the United States Naval Radiological Defence Laboratory exceeds 6 and so provided there are no physiological and physicochemical scale effects, the assembly can be regarded as satisfactory for all heating times. The value of hR for the assembly at Rochester University (9), (10) would, however, appear to be too low for complete equivalence of disk and plane but so long as the heating times are less than 100 seconds the thermal error in scaling is at the most about 10 per cent. For the two assemblies at the Joint Fire Research Organization (11), (12) the scaling is satisfactory only as long as the heating times are restricted to $2\frac{1}{2}$ seconds for the searchlight and 25 seconds for the tungsten lamp when oak is used. On this latter assembly the scaling with lighter materials would probably be satisfactory for all heating times.

7. References

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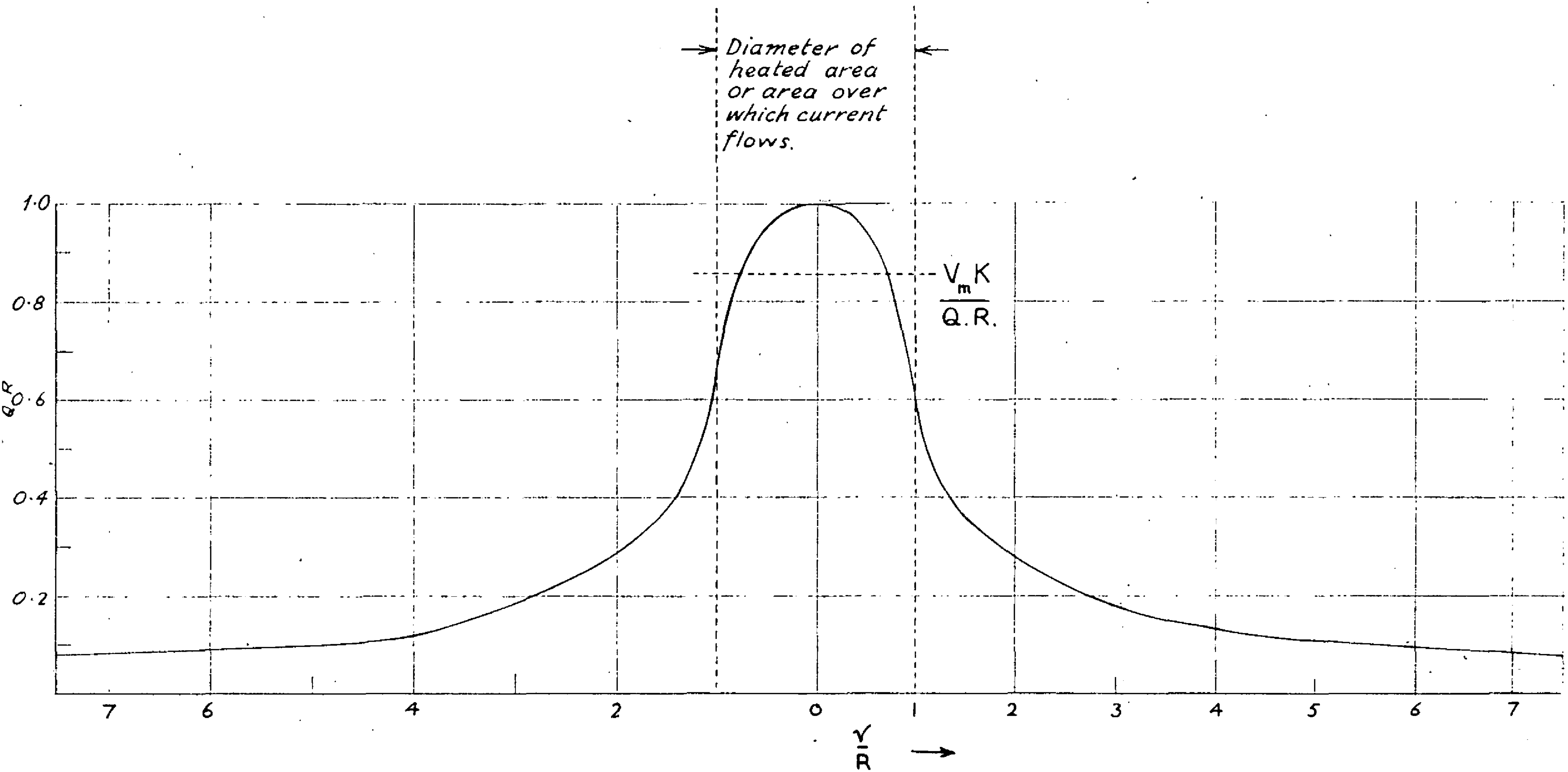

FIG. 1. SURFACE TEMPERATURE DUE TO UNIFORM DISK OVER $0 < r < R$.

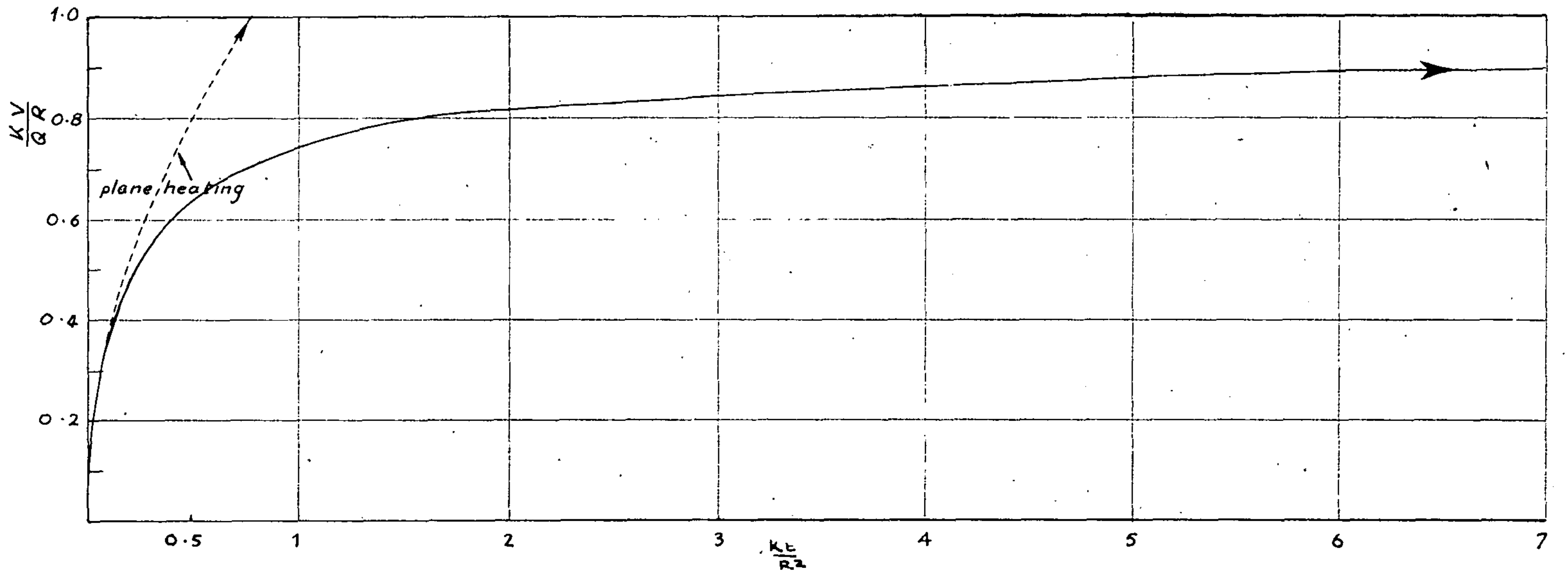

FIG. 2. THE TEMPERATURE ON THE CENTRE OF A CIRCULAR DISK HEATED BY A UNIFORM FLUX (AFTER BLOK)

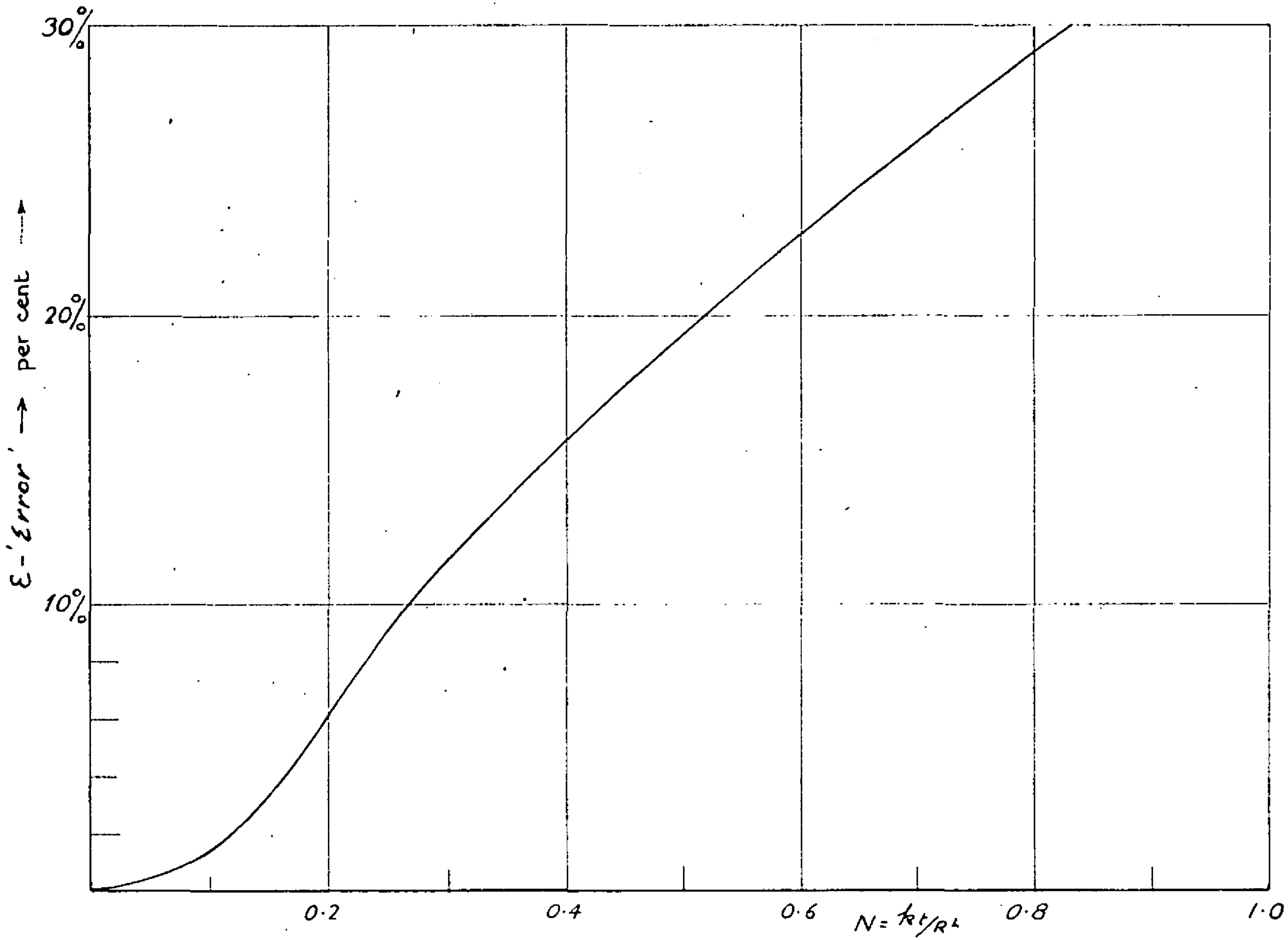

FIG. 3. THE DIFFERENCE BETWEEN THE DISK AND PLANE AS A FRACTION OF THE PLANE TEMPERATURE AT VARIOUS TIMES.

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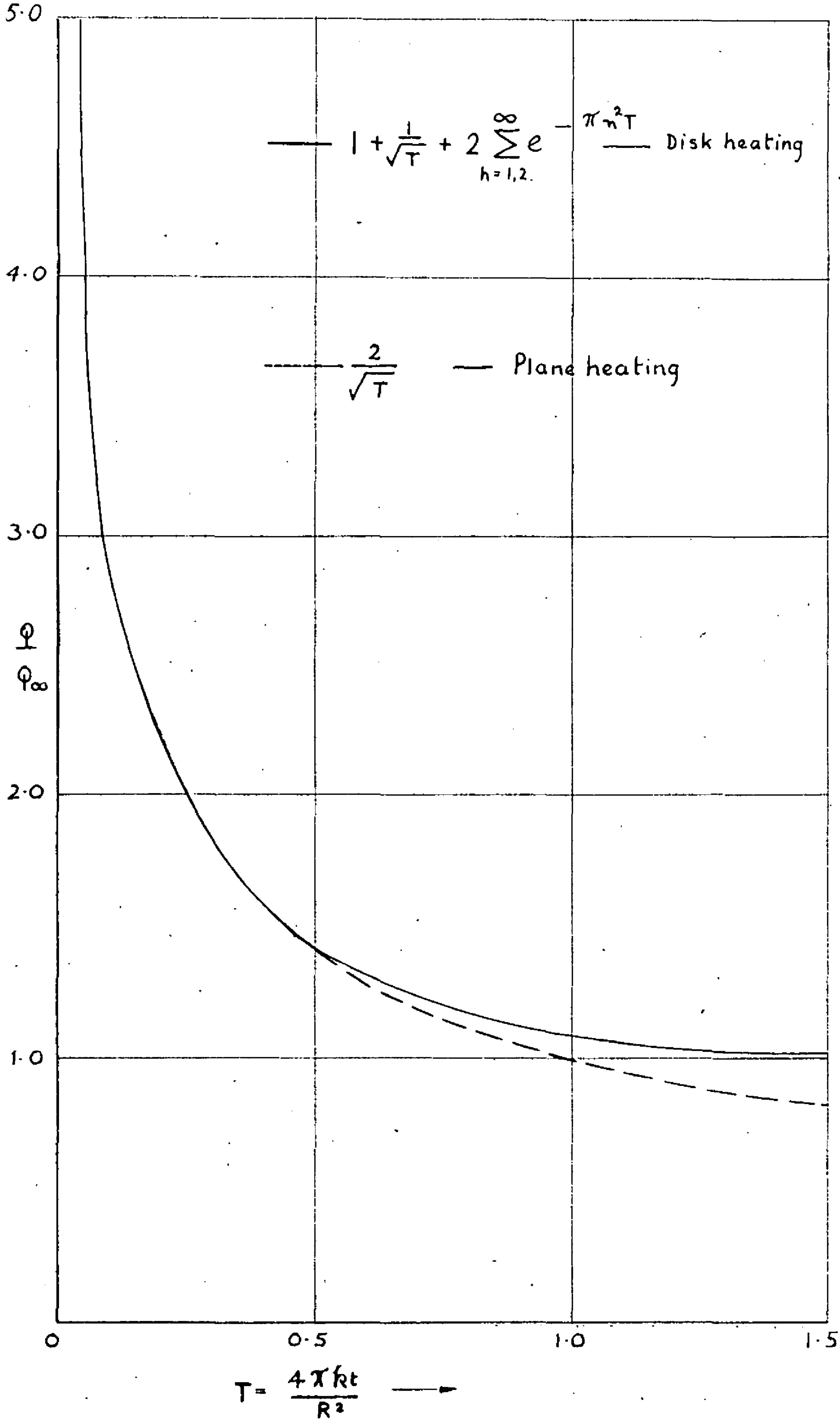

FIG. 5. *THE FLUX REQUIRED TO MAINTAIN A CONSTANT INCREASE OF TEMPERATURE.*

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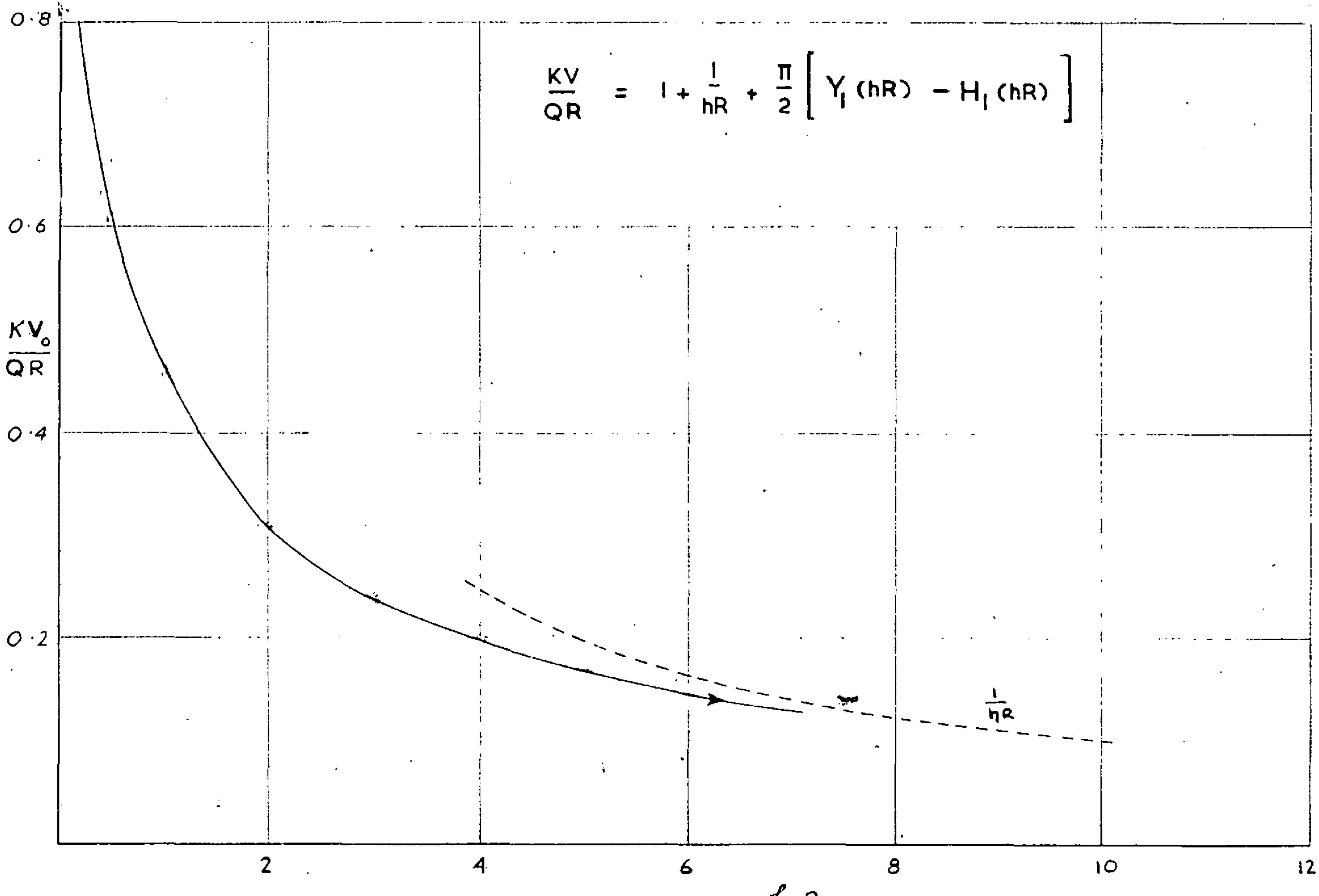

FIG. 6. THE STEADY TEMPERATURE OF THE CENTRE OF A DISK WITH SURFACE COOLING.

F.R.163. 1/1534