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THE REPRESENTATION OF DISTRIBUTED RESISTANCE AND SHUNT CAPACITANCE CIRCUITS BY LUMPED NETWORKS

by

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The temperature θ at any point in a unidirectional heat conduction problem is given by the application of the appropriate boundary conditions to the equation

$$\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2}$$

where k is the thermal diffusivity of the material considered.

The potential difference across a transmission line with distributed series resistance and shunt capacity is given by the application of the appropriate boundary conditions to the equation

$$\frac{\partial v}{\partial t} = \frac{1}{RC} \frac{\partial^2 v}{\partial t^2}$$

The similarity between the two equations makes it apparent that the solution of a heat conduction problem may be obtained by solving the analogous electrical problem. It is however not usually practical to construct electrical transmission lines in which the capacity is uniformly distributed and consideration must be given to the representation of such a line by lumped networks. The representation of distributed networks with boundary conditions involving a potential difference have already been discussed. The present note relates to problems in which the boundary flux is specified.

Comparison of circuits with distributed or lumped parameters

The problem of transmission along a line with distributed resistance and capacity is well known (a particular case is the submarine cable problem) and the potential difference across the circuit at a distance x from the sending end at a time t after a current I_o is fed into the line from a generator having an infinite impedance is given by

$$V_x = 2I_oR\left[\left(\frac{t}{\pi RC}\right)^{\frac{1}{2}} e^{-\frac{x^2RC}{4t}} - \frac{x}{2}\, \mathrm{erfc}\, \frac{x}{2}\sqrt{\frac{RC}{t}}\right] \quad \ldots\ldots.(1)$$

where R and C are the resistance and capacitance per unit length. Corresponding expressions may be derived (Appendix I) for equivalent networks of both T and π sections. These are:

T section

$$V_r = I_oR'\left[\int_0^\lambda e^{-\lambda} I_r(\lambda)\, d\lambda + \frac{1}{2} e^{-\lambda} I_r(\lambda)\right] \quad \ldots\ldots.(2)$$

π section

$$V_r = I_oR'\int_0^\lambda e^{-\lambda} I_r(\lambda)\, d\lambda \quad \ldots\ldots.(3)$$

Where Vr is the voltage across the network at the rth section.

R' and C' are the resistance and capacitance associated with each section.

$I_r(\lambda)$ is the modified Bessel function of the rth order.

and $\lambda = \frac{2t}{R'C'}$

If it is desired to compare the voltage at a distance x along an infinite length of uniform transmission line with that given by lumped circuits, R' must be replaced by αR where α is the length of smooth line represented by one section of the network. Similarly $C' = \alpha C$ and of course $x = r\alpha$. Making these substitutions equations (1) (2) and (3) become:-

Smooth line

$$V_x = I_0 R \left\{ \left[\frac{2\lambda}{\pi}\right]^{\frac{1}{2}} e^{-\frac{x^2}{2\lambda}} - x \operatorname{erfc} \frac{x}{\sqrt{2\lambda}} \right\} \qquad \ldots\ldots.(4)$$

$$\text{where } \lambda = \frac{2t}{RC}$$

T section network

$$V_x = I_0 R\alpha \left[\int_0^{\gamma} I_{\frac{x}{\alpha}}(\gamma)\, d\gamma + \tfrac{1}{2} e^{-\gamma} I_{\frac{x}{\alpha}}(\gamma) \right] \qquad \ldots\ldots.(5)$$

where $\gamma = \frac{2t}{\alpha^2 RC} = \frac{\lambda}{\alpha^2}$ and $\frac{x}{\alpha}$ will have integral values corresponding to the number of sections under consideration.

π section network

$$V_x = \alpha I_0 R \int_0^{\gamma} e^{-\gamma} I_{\frac{x}{\alpha}}(\gamma)\, d\gamma \qquad \ldots\ldots.(6)$$

In comparing the voltage on smooth lines with that of equivalent points (x) on networks it will be seen that the differences depend on the three variables x, α and λ representing, respectively, the distance along the network, the smoothness of the network and the time at which the measurement is made. The networks represent the smooth line least well when they are presented with a discontinuous current function, that is at the beginning of the line, and so any enquiry into the ability of the network to represent a smooth line must deal with this condition.

The difference in potential at the beginning of either a T or π section filter and a smooth network may be expressed by making appropriate substitutions in (4) in the form

$$\alpha I_0 R f\left(\frac{\lambda}{\alpha^2}\right) \text{ and}$$

this means that the maximum error will thus be also proportional to α and the time at which it occurs will be reduced proportionately to α^2. The maximum error is thus inversely proportional to the number of sections representing unit length of smooth network. The fractional error between T or π and smooth networks is of the form $\phi\left(\frac{\lambda}{\alpha^2}\right)$

This is independent of the number of sections per unit length but the time of the occurrence of the maximum will be reduced with the square of the number of sections representing unit length of smooth network.

The absolute and fractional errors for T and π sections are shown in Fig. 1 and 2 respectively.

The number of sections required to ensure that the error is within a specified fraction is shown as a function of $\lambda = {}^{2t}/_{RC}$ in Fig. 3 for both T and π sections where it will be seen that there is little to choose between the performance of the networks for the values chosen.

The performance of semi-infinite smooth and lumped networks at points at any distance from the sending end may be predicted from dimensional considerations, and thus it may be shown that the fractional error at any point between smooth and lumped network composed of either T or π sections will be a function of x/α and γ.

This may readily be shown from expressions 4, 5 and 6. Therefore if a lumped line is constructed so that it gets progressively coarser with distance (α varies x) the fractional error will remain constant at times increasing as α^2. It may be also shown that the same conditions would apply if the voltage error were computed as a fraction of the voltage at the sending end in the network.

REFERENCES

1. LAWSON, D. I. and McGUIRE, J. H. "The solution of transient heat-flow problems by analogous electrical networks". Proc. Instn. Mech. Engrs, A, 1953, 167 (3) 275 - 90.

APPENDIX I

Voltage along a repeated network when a constant current is applied

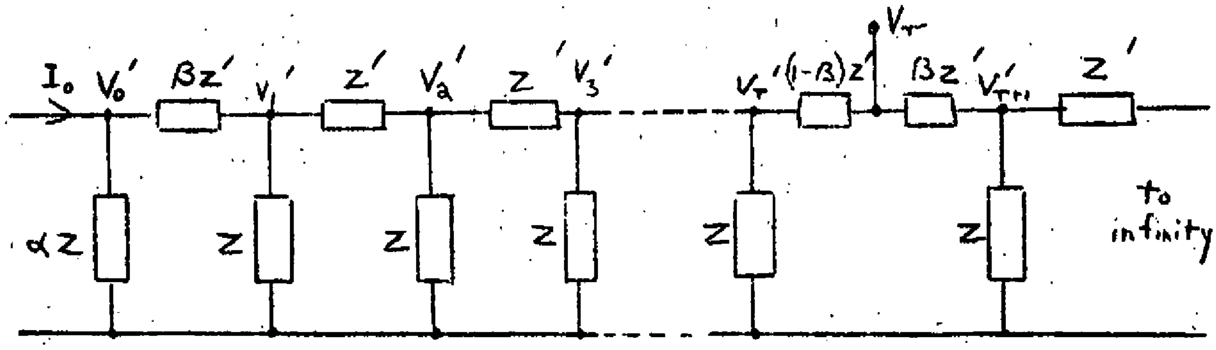

Fig 4 semi-infinite repeated network

A semi-infinite repeated network is shown in fig 4. Since there is no accumulation of change at the interconnecting points the following operational relation holds

$$\frac{\overline{V_r'} - \overline{V_{r+1}'}}{\overline{Z}'} = \frac{\overline{V_{r+1}'}}{\overline{Z}} + \frac{\overline{V_{r+1}'} - \overline{V_{r+2}'}}{\overline{Z}'}$$

hence $\overline{V_r'} - 2\overline{V_{r+1}'}\delta + \overline{V_{r+2}'} = 0$ where $\delta = 1 + \frac{\overline{Z}'}{2\overline{Z}}$ ....(1)

The above expression is valid for $r \geqslant 1$

A solution will exist of the form

$$\overline{V_r'} = A\mu^r$$

where A will be determined by the boundary conditions. Substituting in (1)

$$\mu = \delta \pm \sqrt{\delta^2 - 1}$$

Since the signal is continuously attenuated the positive root may be neglected.

Substituting for $\overline{V_1'}$ and $\overline{V_2'}$ in the first and second meshes it may be shown that:-

$$A = \frac{\alpha \overline{Z}' \overline{I_0}}{(\delta-1)\left[2\mu + (1-\mu)\left(\frac{\alpha}{\delta-1} + 2\beta\right)\right]}$$

and hence $$\overline{V_r'} = \frac{\alpha \overline{Z}' \overline{I_0} \mu^r}{(\delta-1)\left[2\mu + (1-\mu)\left(\frac{\alpha}{\delta-1} + 2\beta\right)\right]} \quad ....(2)$$

From fig 4 it is seen that

$$\overline{V_r} = \beta\overline{V_r'} + (1-\beta)\overline{V_{r+1}'} \quad ....(3)$$

Substituting for $\overline{V_r'}$ in (3) gives

$$\overline{V_r} = \frac{\alpha \overline{Z}' \overline{I_0}\left[\beta + (1-\beta)\mu\right]\mu^r}{(\delta-1)\left[2\mu + (1-\mu)\left(\frac{\alpha}{\delta-1} + 2\beta\right)\right]}$$

APPENDIX I (continued)

Well known cases occur when

$\alpha = 0$ and $\beta = \frac{1}{2}$ — T section filters

$\alpha = 2$ and $\beta = 1$ — π section filters

when

$$\overline{V_r} = \overline{I_0}\,\frac{Z}{2}\,\frac{1+\mu}{1-\mu}\,\mu^r \qquad \text{T section} \qquad \ldots\ldots(4)$$

$$\overline{V_r} = \frac{\overline{I_0} Z' \mu^r}{\delta - \mu} \qquad \pi \text{ section} \qquad \ldots\ldots(5)$$

Inverting (4) and (5) gives

$$V_r = \frac{I_0 R}{2}\left[2\int_0^\lambda e^{-\lambda} I_r(\lambda)\,d\lambda + e^{-\lambda} I_r(\lambda)\right] \qquad \text{T section} \qquad \ldots\ldots(6)$$

$$V_r = I_0 R \int_0^\lambda e^{-\lambda} I_r(\lambda)\,d\lambda \qquad \pi \text{ section} \qquad \ldots\ldots(7)$$

where $\lambda = kt = \frac{2t}{RC}$

APPENDIX II

Comparison of π and T sections networks with a smooth transmission line

Let each section of the iterated network represent a length α of the smooth line having primary constants R and C. The individual components of the network will have therefore values αR and αC. The voltage after r sections will be as in Eqn.(6) and Eqn.(7) of Appendix I except that λ will have the value $2t/\alpha^2RC$

It may be shown that the voltage V_x at a distance x along a smooth line to which is applied a constant current I_0 at the time t = o is given by

$$V_x = 2I_oR\left\{\left(\frac{t}{\pi RC}\right)^{\frac{1}{2}} e^{-\frac{x^2RC}{4t}} - \frac{x}{2}\,\mathrm{erfc}\,\frac{x}{2}\sqrt{\frac{RC}{t}}\right\}$$

The rth section of the interated network will represent a distance $x = r\alpha$ along the smooth line

thus

$$V_r = I_oR\left\{\alpha\sqrt{\frac{2\lambda}{\pi}}\, e^{-\frac{r^2}{2\lambda}} - r\alpha\,\mathrm{erfc}\,\frac{r}{\sqrt{2\lambda}}\right\} \qquad \text{.......(8)}$$

In comparing the smooth line with the T and π networks the following expressions must be evaluated.

$$I_oR\left\{\alpha\sqrt{\frac{2\lambda}{\pi}}\, e^{-\frac{r^2}{2\lambda}} - r\alpha\,\mathrm{erfc}\,\frac{r}{\sqrt{2\lambda}}\right\} \qquad \text{......smooth line.}$$

$$I_oR\alpha\left\{\int_0^\lambda e^{-\lambda} I_r(\lambda)\,d\lambda + \frac{e^{-\lambda}}{2} I_r(\lambda)\right\} \qquad \text{.......T section}$$

$$I_oR\alpha\int_0^\lambda e^{-\lambda} I_r(\lambda)\,d\lambda \qquad \text{.......}\pi\text{ section}$$

These expressions may be made to refer to unit length of network by imposing the restriction $r\alpha = 1$ and the difference between the result given by smooth network and the T and π sections is shown in figs. 1 and 2 .

The maximum difference between the correct results given by a smooth line and T and π networks is shown as a function of the number of sections per unit length in fig. 3 .

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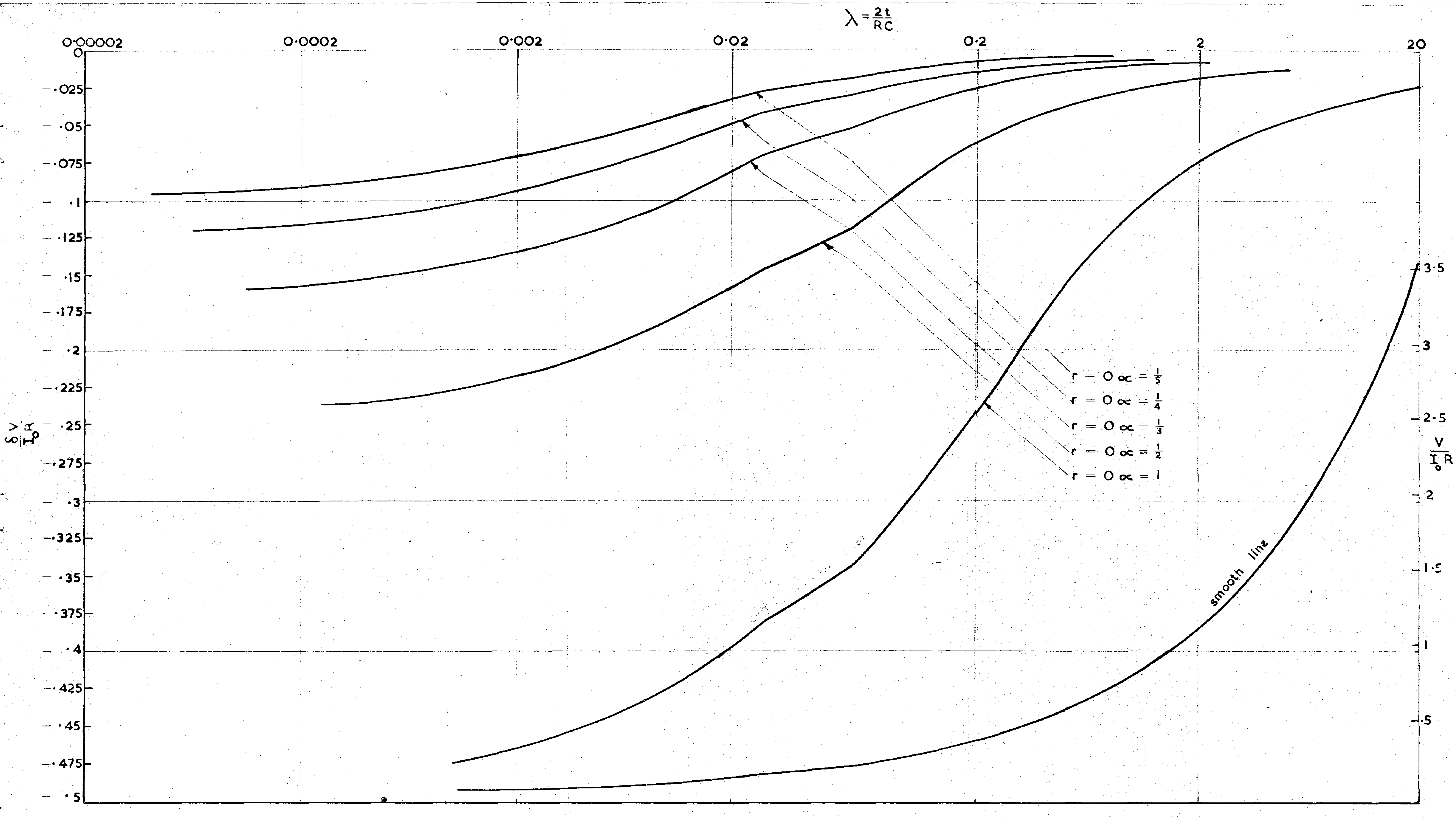

FIG. 1a. THE DIFFERENCE AT $x = 0$ BETWEEN THE POTENTIALS ACROSS T SECTION AND SMOOTH LINES

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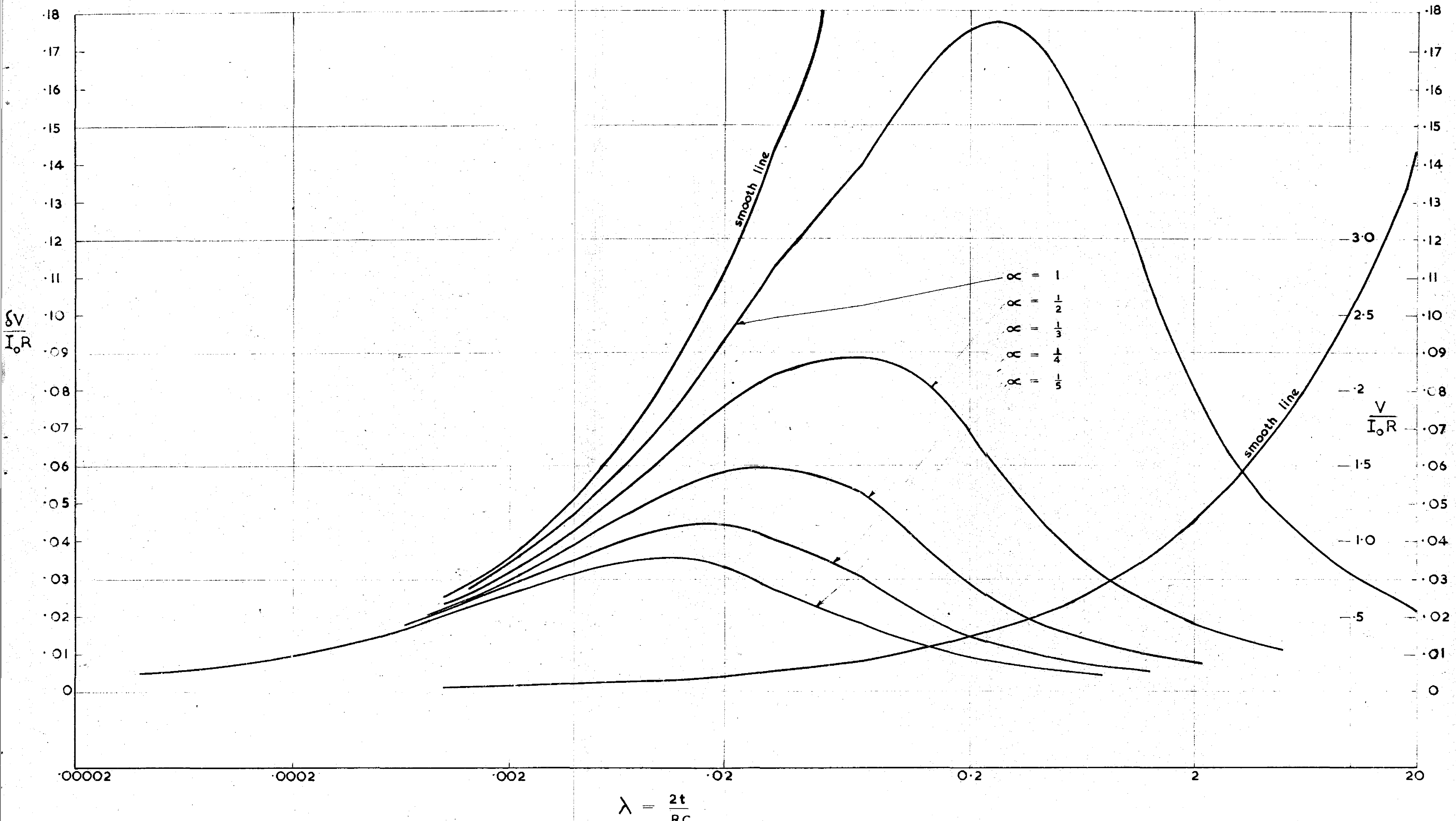

FIG. 1b. THE DIFFERENCE AT $x = 0$ BETWEEN THE POTENTIALS ACROSS Π SECTION AND SMOOTH LINES

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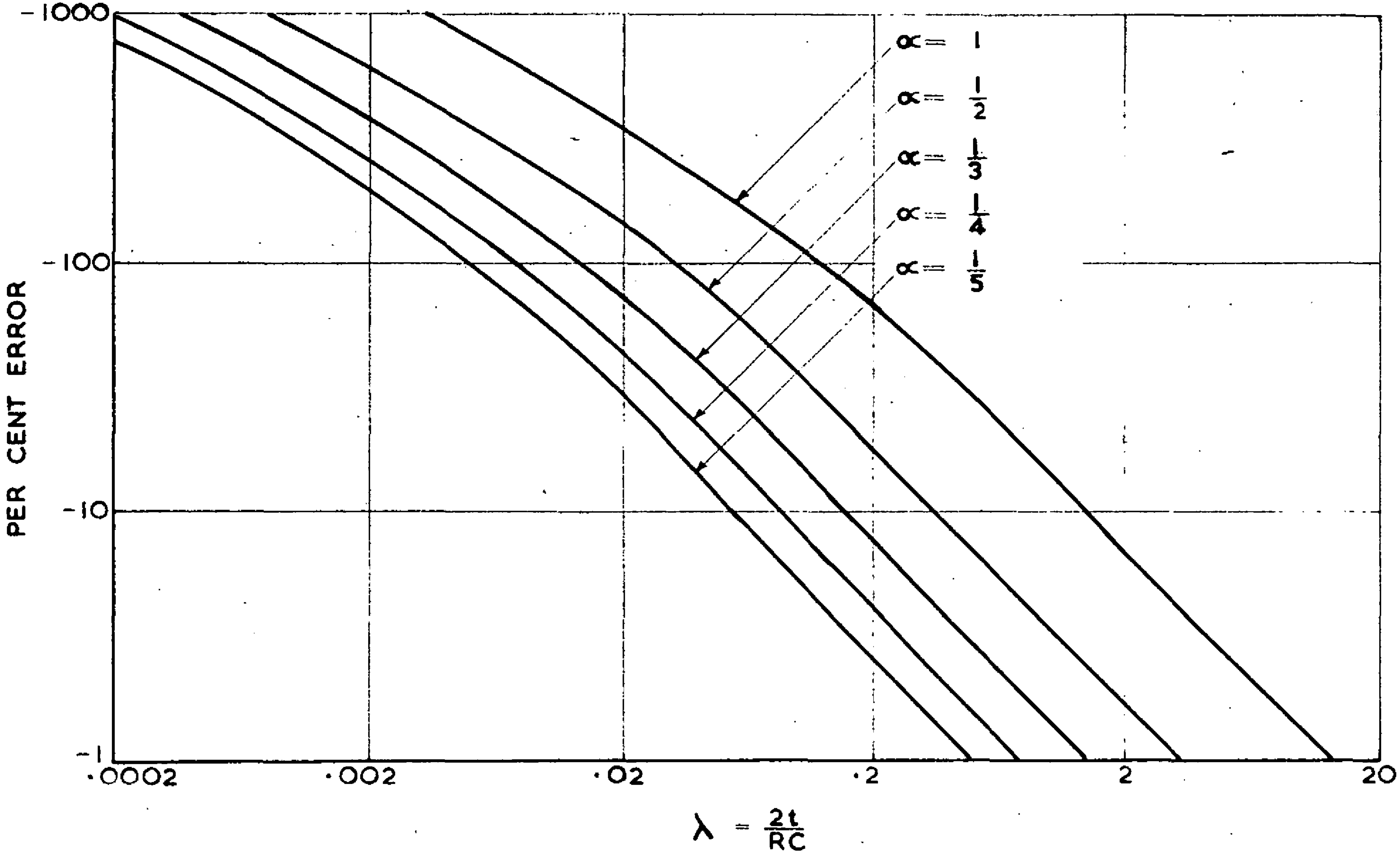

FIG. 2a. THE DIFFERENCE AT $x = 0$ BETWEEN THE POTENTIALS ACROSS T SECTION AND SMOOTH LINES EXPRESSED AS A FRACTION OF THE POTENTIAL DIFFERENCE ACROSS THE SMOOTH LINE

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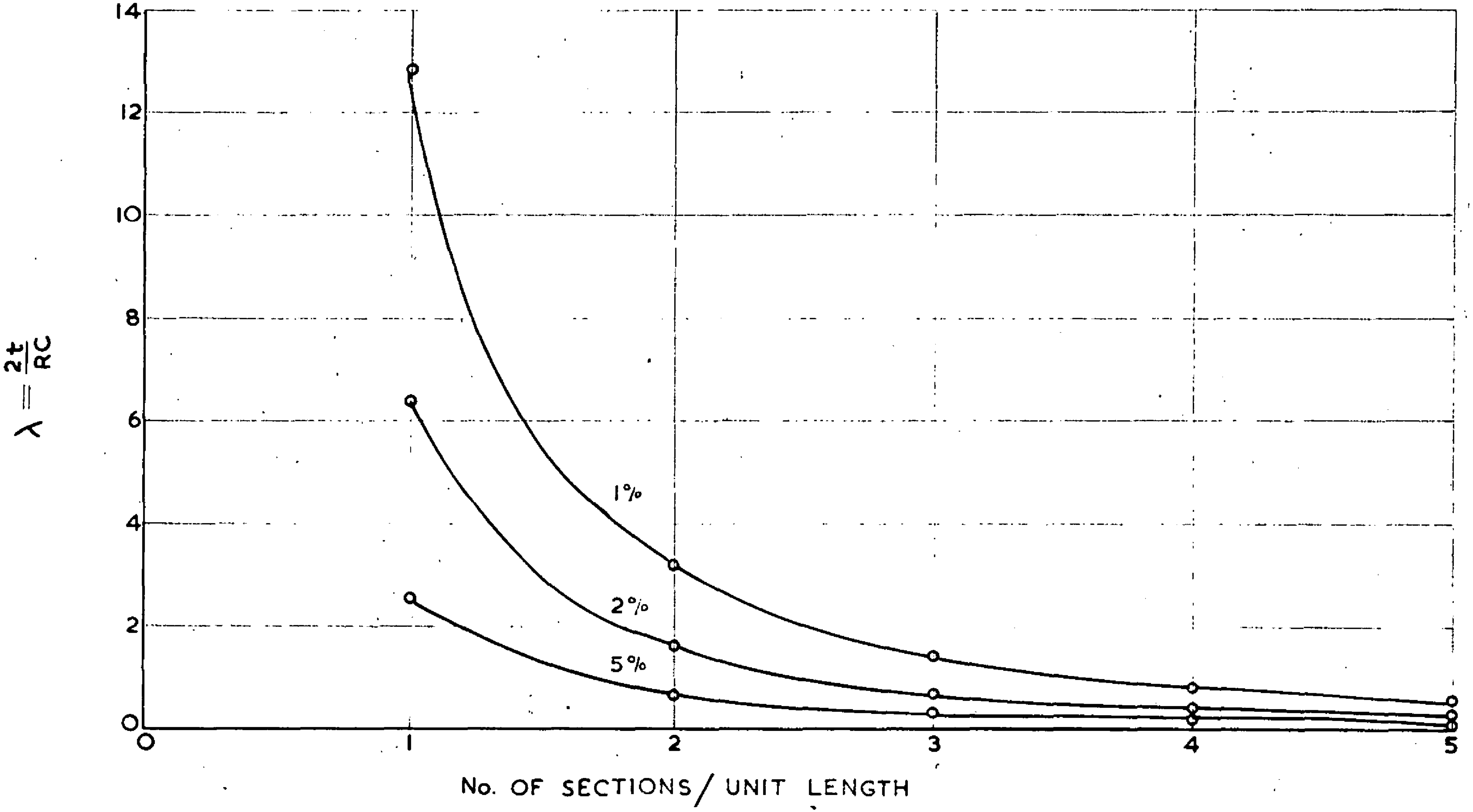

FIG. 3a. THE NUMBER OF SECTIONS PER UNIT LENGTH (FOR A T SECTION LINE) REQUIRED TO REPRESENT A SMOOTH LINE TO A GIVEN ACCURACY

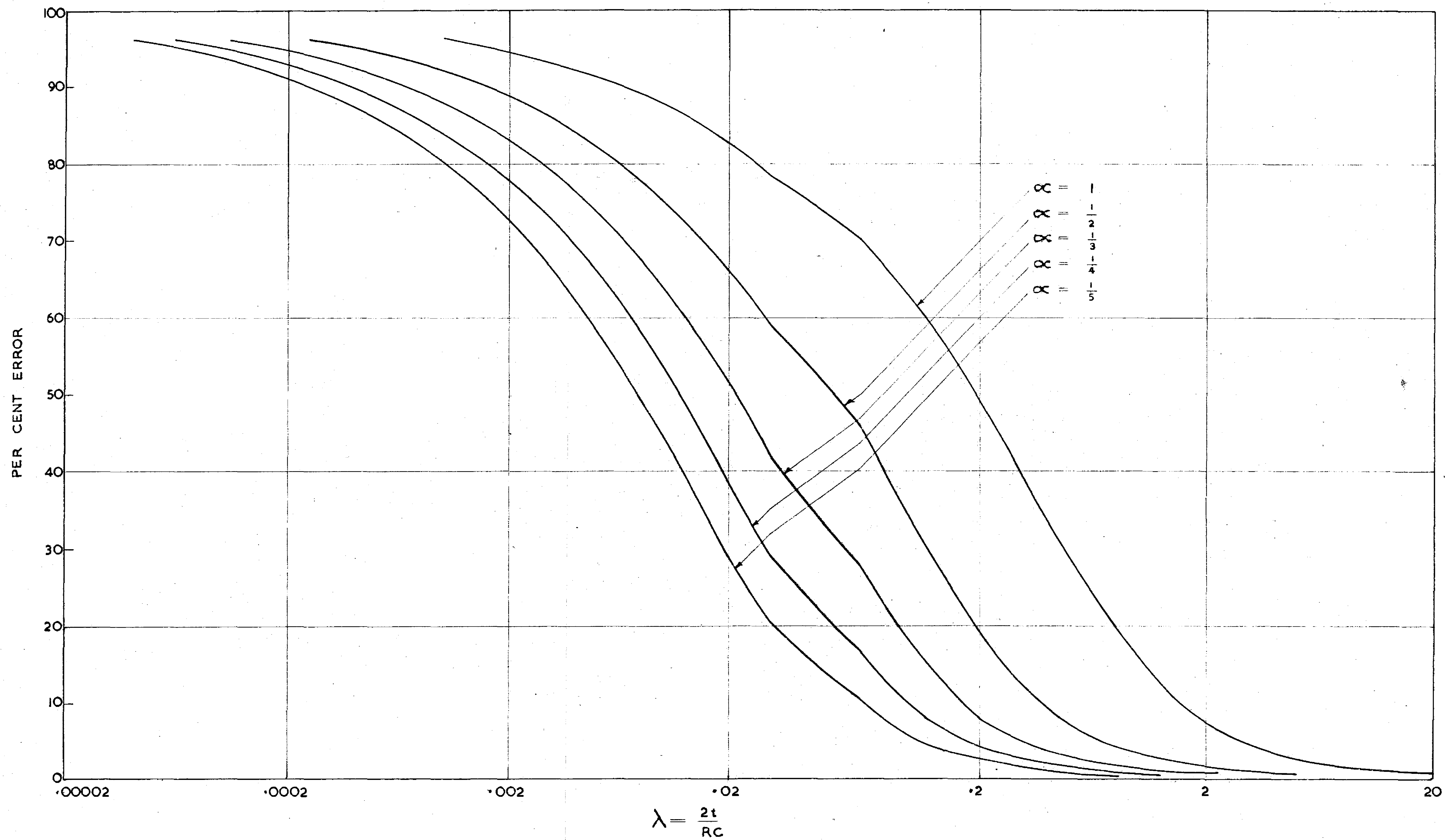

FIG.2b. THE DIFFERENCE AT $x=0$ BETWEEN THE POTENTIALS ACROSS π SECTION AND SMOOTH LINES EXPRESSED AS A FRACTION OF THE POTENTIAL DIFFERENCE ACROSS THE SMOOTH LINE

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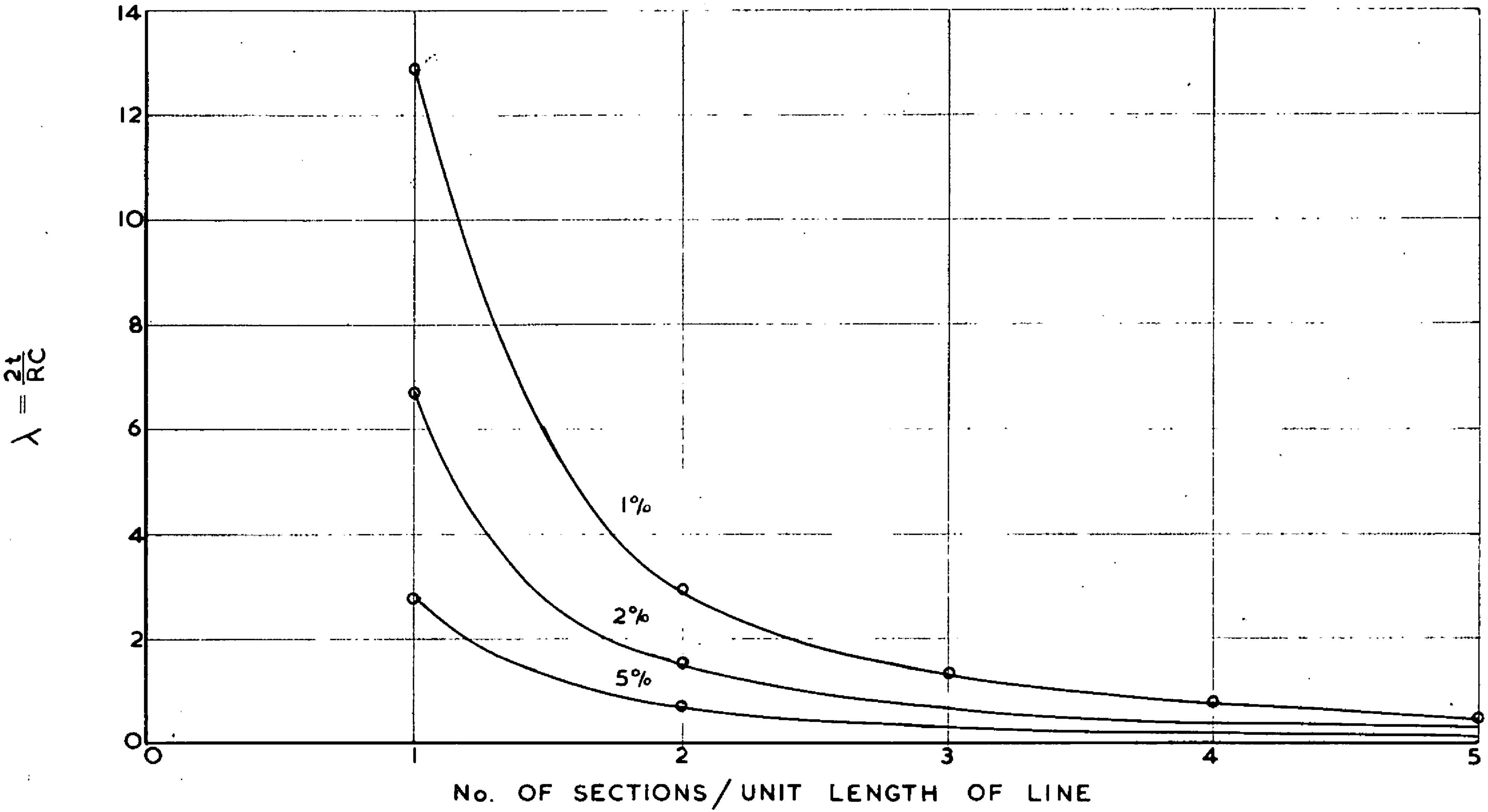

FIG. 3b. THE NUMBER OF SECTIONS PER UNIT LENGTH (FOR A Π SECTION LINE) REQUIRED TO REPRESENT A SMOOTH LINE TO A GIVEN ACCURACY