F.R. Note No. 197/1955
Research Programme
Objective B/5

DEPARTMENT OF SCIENTIFIC AND INDUSTRIAL RESEARCH AND FIRE OFFICES' COMMITTEE
JOINT FIRE RESEARCH ORGANIZATION

SELF HEATING IN CONDUCTING SLABS, CYLINDERS AND SPHERES WITH NEWTONIAN SURFACE COOLING

by

P. H. Thomas

Summary

The analysis given by Chambré is extended to include the problem of the self heating for bodies generating heat according to the Arrhenius law with Newtonian surface cooling; Chambrés results appearing as particular cases for infinite surface cooling.

July, 1955.

Fire Research Station,
Boreham Wood,
Herts.

SELF HEATING IN CONDUCTING SLABS, CYLINDERS AND SPHERES WITH NEWTONIAN SURFACE COOLING

by

P. H. Thomas

Introduction

The analysis described below was undertaken as part of a programme of work into the self heating of wood carried out at the Joint Fire Research Organization. The generation of heat is assumed to follow the Arrhenius law for a monomolecular reaction. This theory provides a basis for studying experimental data and the extent to which other factors such as the pressence of a secondary reaction and exhaustion of combustible material are important. The analysis described below closely follows Chambrés [1] method.

Chambrés notation is followed i.e.

T_o		is absolute ambient temperature
T		is absolute temperature
x	=	distance coordinate
r	=	radius of sphere, cylinder, or half width of slab
z	=	x/r
E	=	energy of activation of reaction
R	=	universal gas constant
λ		is thermal conductivity
Q	=	heat of reaction
l		a constant

In addition we have H the cooling coefficient at the surface.

Mathematically the steady state problem can be represented by

$$\nabla^2 T = -\frac{Q}{\lambda} l \, e^{-E/RT} \qquad \ldots\ldots (1)$$

where ∇^2 is the Laplacian operator.

For small temperature differences we can write

$$\frac{-E}{RT} \doteqdot -\frac{E}{RT_o}\left(1 - \frac{T - T_o}{T_o}\right)$$

Denoting

$$\theta = \frac{E}{RT_o^2}(T - T_o) \qquad \ldots\ldots (2)$$

we have

$$\frac{d^2\theta}{dz^2} + \frac{k}{z}\frac{d\theta}{dz} = -\delta e^{\theta} \qquad \ldots\ldots (3)$$

where k = 0 for a slab
= 1 for a cylinder
= 2 for a sphere

& $$\delta = \frac{Q}{\lambda} l . r^2 \frac{E}{RT_o^2} e^{-E/RT_o} \qquad \ldots\ldots (4)$$

The boundary conditions are

$$\text{At} \quad Z = 0 \qquad \frac{d\theta}{dZ} = 0 \quad \text{by symmetry} \qquad \ldots\ldots (5)$$

At $Z = 1$

$$\frac{Hr}{\lambda}\theta + \frac{d\theta}{dZ} = 0 \qquad \ldots\ldots (6)$$

We denote $\frac{Hr}{\lambda}$ by the dimensionless parameter α

It is required to find the temperature distribution in the body and in particular the maximum value of 'δ' for which such a solution is possible. For 'δ' in excess of this critical value δ_c no steady state solution exists and the material will eventually ignite or explode.

Slab

This problem has been investigated in more detail than for the cylinder and the sphere but for the sake of completion the solution is given here. Frank - Kamensky [2] and Harris [3] have obtained the solution for $\alpha \rightarrow \infty$.

The equation with 'k' zero may be integrated (multiply by $\frac{d\theta}{dZ}$ and integrate) and the solution obtained as

$$\theta = \log A - 2\log_e \cosh Z\sqrt{\frac{\delta A}{2}} \qquad \ldots\ldots (7)$$

where A is a constant of integration, the logarithm of which gives the temperature at the centre. This satisfies the first boundary condition (5) and satisfies (6) if

$$\log_e \delta = \log_e 2D^2 - 2\log_e \cosh D - \frac{2D\tanh D}{\alpha} \ldots (8)$$

$$\text{where} \quad D = \sqrt{\frac{\delta A}{2}}$$

D may be found by the condition that 'δ' is to be a maximum, thus differentiating equation (8)

$$\alpha = \frac{D\sinh D\cosh D + D^2}{(1 - D\tanh D)\cosh^2 D} \qquad \ldots\ldots (9)$$

For any value of D less than the root of $1 - D \tanh D = 0$
α can be calculated from (9).

For these values of D, δ_c can be found from equation (8). 'A' can then be calculated and so the temperature distribution, in particular the centre and surface temperatures, θ_0 & θ_s respectively obtained as functions of α . The results are given graphically in Figures (1), (2), (3) and (4).

Cylinder

For $k = 1$, following Chambré, the solution is

$$\theta = \operatorname{Log}_e \frac{8B/\delta}{(Bz^2+1)^2} \qquad \ldots\ldots (10)$$

where B is a constant of integration.

The boundary condition (6) is satisfied if

$$\alpha \operatorname{Log}_e \frac{8B/\delta}{(B+1)^2} = \frac{4B}{B+1} \qquad \ldots\ldots (11)$$

$$\therefore \operatorname{Log}_e \delta = \operatorname{Log} \frac{8B}{(B+1)^2} - \frac{4B}{\alpha(B+1)} \qquad \ldots\ldots (12)$$

The critical value of 'δ' is obtained by putting

$$\frac{d\delta}{dB} = 0$$

from which is obtained

$$\alpha = \frac{4B}{1-B^2} \qquad \ldots\ldots (13)$$

Thus in terms of a critical value of B given by (13) we have from (12) and (13)

$$\delta_c = \frac{8B_c}{(B_c+1)^2} \epsilon^{-(1-B_c)} \qquad \ldots\ldots (14)$$

from (10) and (14)

$$\theta_0 = 2\operatorname{Log}_e(1+B_c) + 1 - B_c \qquad \ldots\ldots (15)$$

and $$\theta_s = 1 - B_c \qquad \ldots\ldots (16)$$

These parameters are shown as functions of α in Figures (1), (2), (3) and (4).

Spheres

Following Chambré we define

$$\psi = \theta_0 - \theta \qquad \ldots\ldots (17)$$

where θ_0 is the unknown centre temperature

and $$\zeta = z\left(\delta e^{\theta_0}\right)^{1/2} \qquad \ldots\ldots (18)$$

Hence (3) with $k = 2$ becomes

$$\frac{1}{\zeta^2}\frac{d}{d\zeta}\left(\zeta^2\frac{d\psi}{d\zeta}\right) = e^{-\psi} \qquad \ldots\ldots (19)$$

Equation (6) becomes

$$\alpha\left(\theta_0 - \psi_s\right) = \left(\frac{d\psi}{d\zeta}\right)_s\left(\delta e^{\theta_0}\right)^{1/2} \qquad \ldots\ldots (20)$$

where $$\zeta_s = \left(\delta e^{\theta_0}\right)^{1/2} \qquad \ldots\ldots (21)$$

and the suffix 's' refers to the surface.

Equation (5) becomes

$$\psi = \frac{d\psi}{d\zeta} = 0 \quad \text{at } z = 0 \qquad \ldots\ldots (22)$$

Substituting for θ_0 from equation (21) we write equation (20) as

$$\operatorname{Log}_e \delta = -\psi_s + 2\log\zeta_s - \frac{\zeta_s}{\alpha}\frac{d\psi_s}{d\zeta_s} \qquad \ldots\ldots (23)$$

To find the critical condition we differentiate with respect to ζ_s and put $\frac{d\delta}{d\zeta_s}$ equal to zero.

Hence

$$-\frac{d\psi_s}{d\zeta_s} + \frac{2}{\zeta_s} = \frac{1}{\alpha}\left[\frac{d\psi_s}{d\zeta_s} + \zeta_s\frac{d^2\psi_s}{d\zeta_s^2}\right] \qquad \ldots\ldots (24)$$

From equation (19) we then obtain

$$\alpha = \frac{\zeta_s^2 e^{-\psi_s} - \zeta_s\frac{d\psi_s}{d\zeta_s}}{2 - \zeta_s\frac{d\psi_s}{d\zeta_s}} \qquad \ldots\ldots (25)$$

Now ψ and $\frac{d\psi}{d\zeta}$ are tabulated as functions of ζ for the condition (22). For any ζ_s we can now compute α and this value of ζ_s will satisfy the boundary condition (26) with the condition that 'δ' has a maximum value and α is the value computed by equation (31).

From these values of ζ_s $\frac{d\psi}{d\zeta_s}$ and ψ_s the values of θ_0 θ_s and δ_c can be obtained from equations (17), (21) and (24) for any α. The results are shown in Figures (1), (2), (3) and (8).

Limiting solution for small values of α

It is interesting to consider the form of the solution for $\alpha \to 0$. By analogy with the slab and the cylinder we expect $\theta_0 - \theta_s$ to tend to zero so equation (3) becomes

$$\frac{d^2\theta}{dz^2} + \frac{2}{z}\frac{d\theta}{dz} = -\delta e^{\theta_0} \qquad \ldots\ldots (26)$$

Repeated integration gives with $\frac{d\theta}{dz}$ zero at z zero.

$$\theta = \theta_0 - \frac{\delta e^{\theta_0} z^2}{6} \qquad \ldots\ldots (27)$$

and condition (6) becomes

$$\alpha\left(\theta_0 - \frac{\delta e^{\theta_0}}{3}\right) = \frac{\delta e^{\theta_0}}{3} \qquad \ldots\ldots (28)$$

Treating δ as a function of θ_0 we obtain the condition for maximum δ as

$$\theta_0 = 1 \qquad \ldots\ldots (29)$$

From which

$$\delta_c = \frac{3\alpha}{e} \qquad \ldots\ldots (30)$$

This may be directly compared with the results for the slab and the cylinder where the coefficients of α/e are 1 and 2 respectively.

References

(1) P. L. CHAMBRÉ. J. Chem. Phys. 1952, 20 (11), 1795.
(2) D. A. FRANK - KAMENSKY. Asto Physiochimicen. U.R.S.S. 1942, 16, 357.
(3) E. S. HARRIS. Proc. Roy Soc. (London) A175 (1940) 254.
(4) S. CHANDRESEKAR and G. W. WARES. Astophys J. (1940), 109 551.

PC.

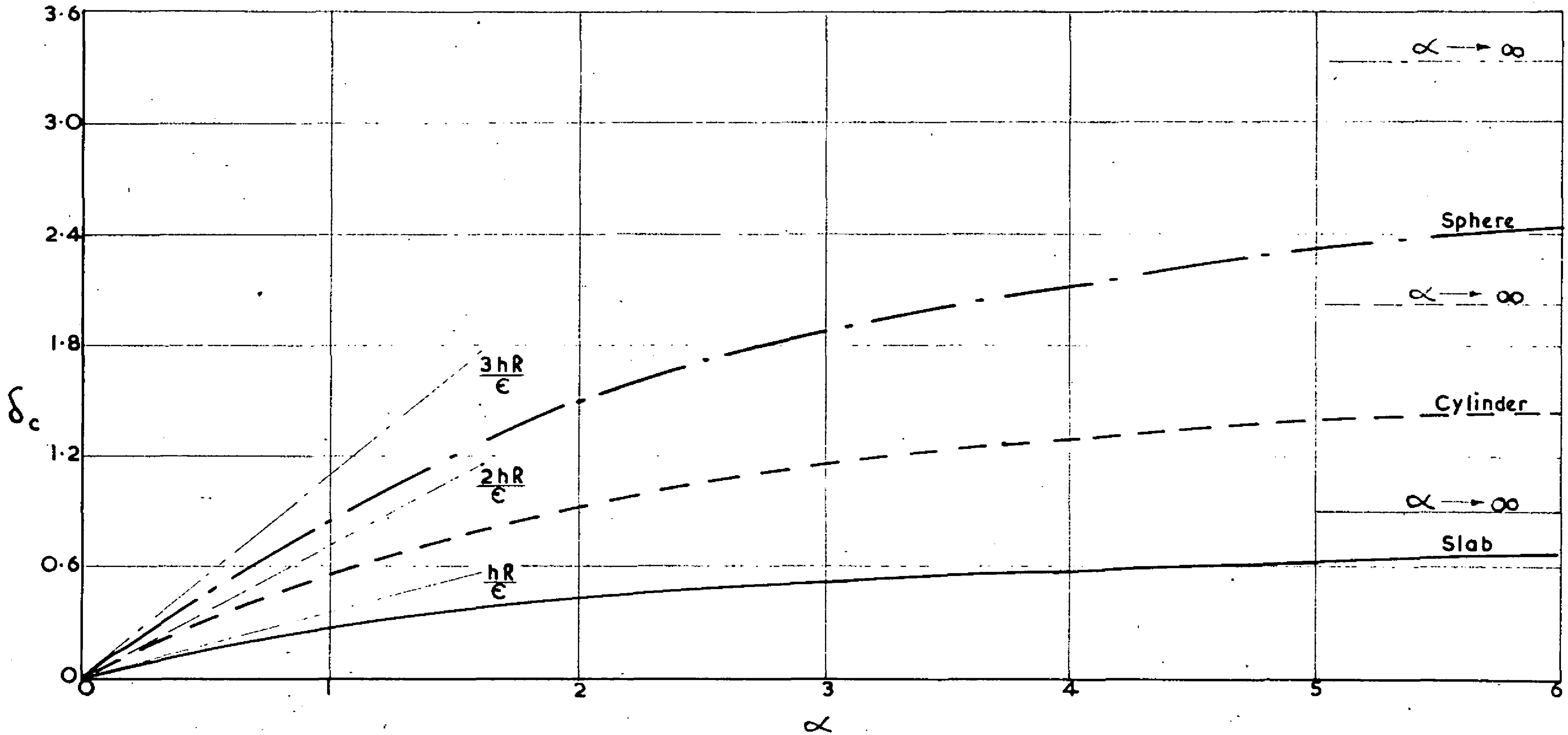

FIG. 1. δ_c AS A FUNCTION OF α FOR A SPHERE, A CYLINDER AND A SLAB

FR 197 1/1706

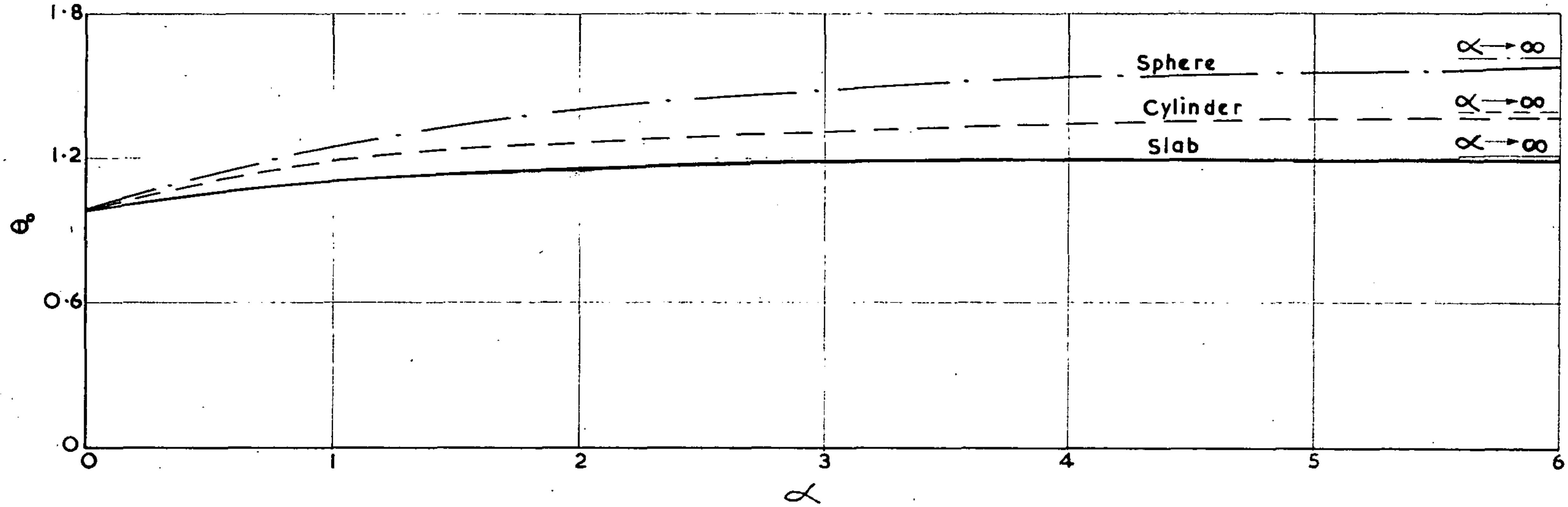

FIG. 2. THE DIMENSIONLESS CRITICAL CENTRE TEMPERATURE θ_0 AS A FUNCTION OF α FOR A SPHERE, A CYLINDER AND A SLAB

F.R. 197 1/1707

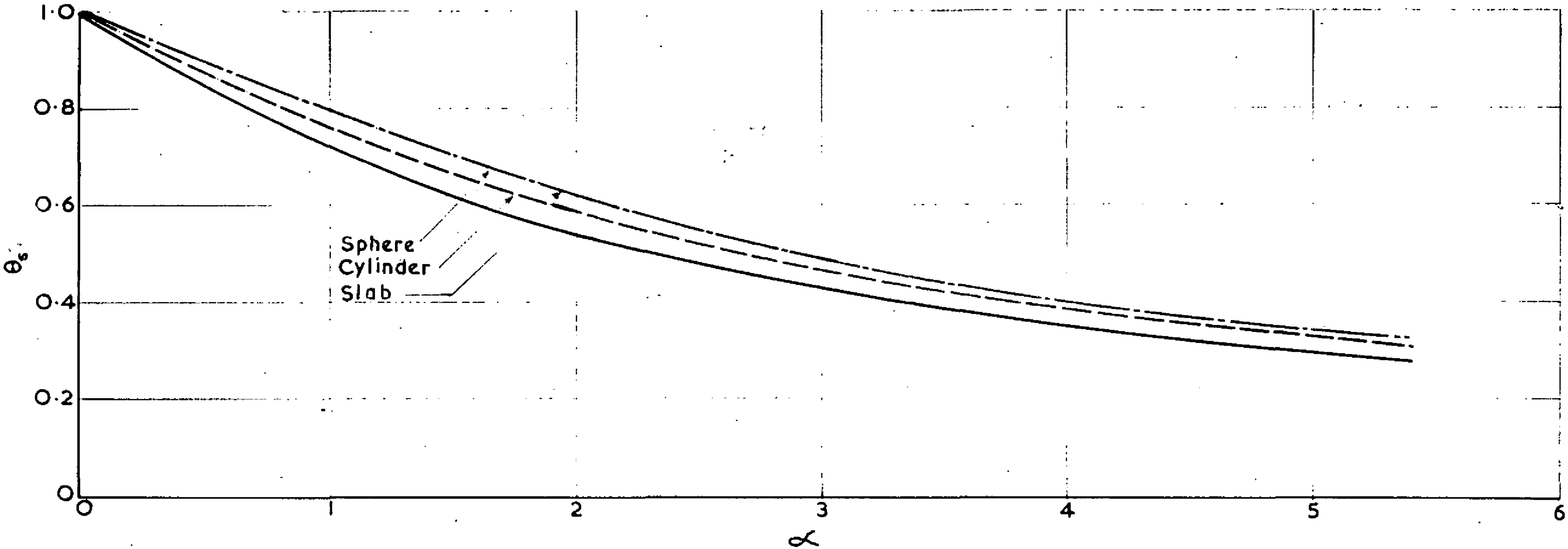

FIG. 3. THE DIMENSIONLESS CRITICAL SURFACE TEMPERATURE Θ_s AS A FUNCTION OF α FOR A SPHERE, A CYLINDER AND A SLAB

F.R. 197. 11/1708

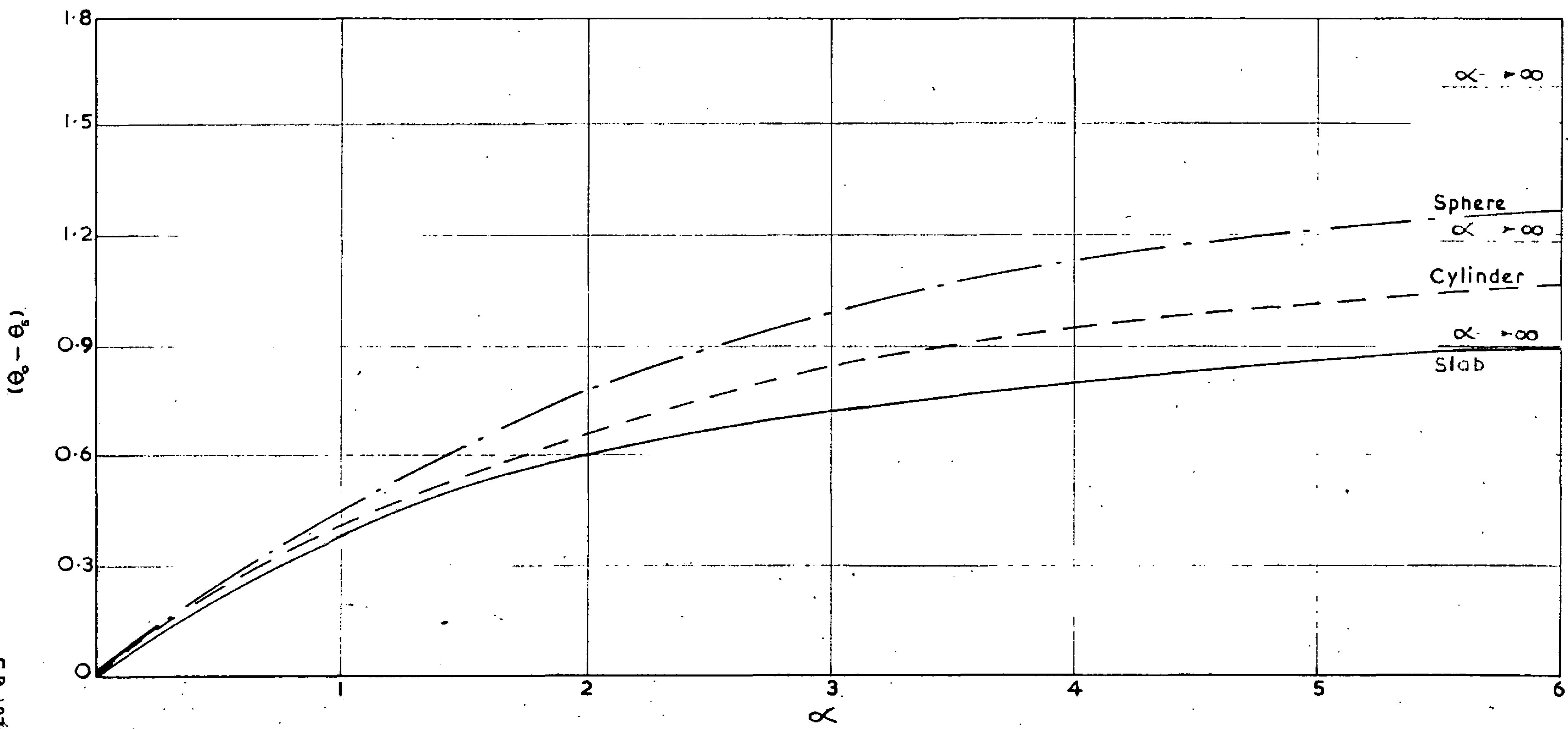

FIG. 4. $\Theta_0 - \Theta_s$ AS A FUNCTION OF α FOR A SPHERE, A CYLINDER AND A SLAB

F.R. 197. F/1709