F.R. Note No. 339/1958

DEPARTMENT OF SCIENTIFIC AND INDUSTRIAL RESEARCH AND FIRE OFFICES' COMMITTEE
JOINT FIRE RESEARCH ORGANIZATION

THE DISTRIBUTIONS OF RADIATION FROM A RADIATING SPHERE AND DISC AT THE FOCUS OF AN ELLIPTIC MIRROR

by

P. H. Thomas and G. C. Karas

Summary

The distribution of radiation at the focus of an elliptical mirror from a sphere or disc at the other focus is calculated and the result compared with the measured distribution for a particular carbon arc system. The values of the maximum intensity are in good agreement but the distribution is narrower than that calculated, presumably as a result of obscuration.

January, 1958.

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1. Introduction

The analysis proceeds in three steps. Firstly (see Fig. 1), a point 'P' on the surface of the mirror is considered. The radiation from the sphere at the focus F_1 striking an elementary area at P produces an elliptically shaped image area in the second focal plane at F_2. The geometry of this having been discussed, the second step is the extension of the elementary area to an annulus through P. The last and final step is the integration over all the area of the mirror. The diameter of the sphere is considered to be small compared with the distance from the mirror. The analysis is repeated for a disc source.

2. Theory

2 (i) The ellipse at the second focus from a point on the mirror

We consider the plane through the major axis of the ellipse and a point P (see Fig. 1).

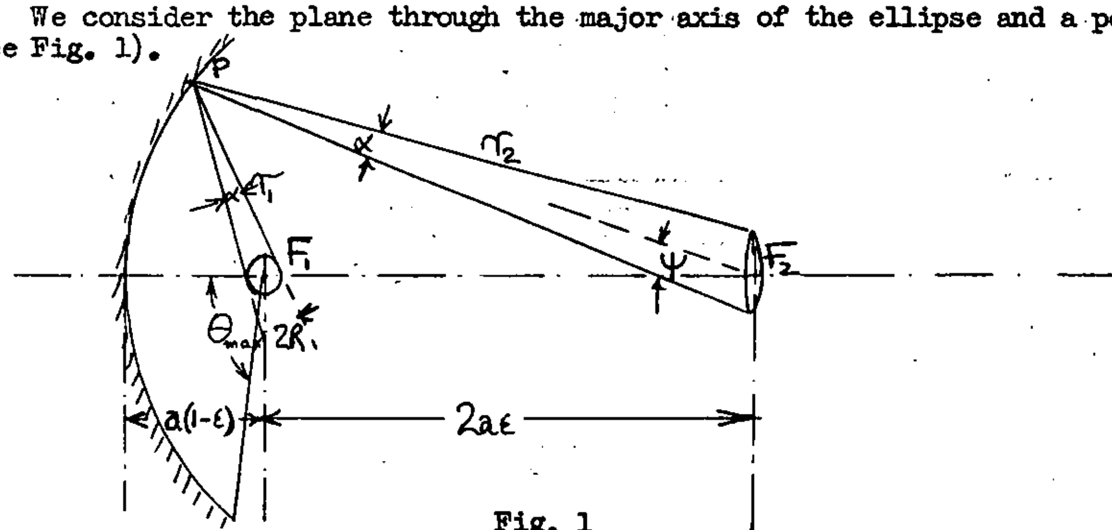

Fig. 1
Geometry of radiation distribution in one plane

Following the notation of Fig. 1, we have from the geometry of the ellipse

$$r_1 = \frac{a\,(1 - \varepsilon^2)}{1 + \varepsilon \cos\theta} \qquad \text{.............. (1)}$$

where ε is the eccentricity of the ellipse
and a is the semi major axis

$$r_2 = 2a - \frac{a\,(1 - \varepsilon^2)}{1 + \varepsilon \cos\theta} \qquad \text{.............. (2)}$$

and
$$F_1 F_2 = 2a\,\varepsilon \qquad \text{.............. (3)}$$

We assume $R << a\,(1 - \varepsilon)$

so that
$$\alpha = \frac{2R}{r_1}$$

which from equation (1) gives

$$\alpha = \frac{2R}{a}\left(\frac{1 + \varepsilon \cos\theta}{1 - \varepsilon^2}\right) \qquad \text{.............. (4)}$$

From the geometry of Δ $F_1 F_2 P$

$$\cos \Psi = \frac{r_2^2 + (F_1 F_2)^2 - r_1^2}{2 r_2 (F_1 F_2)} \qquad \ldots\ldots\ldots\ldots\ldots (5)$$

which from equations (1), (2), and (3) gives

$$\cos \Psi = \frac{2 \varepsilon + (1 + \varepsilon^2) \cos \theta}{(1 + \varepsilon^2) + 2 \varepsilon \cos \theta}$$

image ellipse in the vertical plane at F_2 has a semi-minor axis of B

where $B = \frac{r_2 \alpha}{2}$

which from equation (2) gives

$$B = R \frac{(1 + 2\varepsilon \cos \theta + \varepsilon^2)}{1 - \varepsilon^2} \qquad \ldots\ldots\ldots\ldots\ldots (6)$$

and from equations (5) and (6), a semi-major axis of

$$A = \frac{B}{\cos \Psi} = \frac{R(1 + 2\varepsilon \cos \theta + \varepsilon^2)^2}{((1 + \varepsilon^2) \cos \theta + 2\varepsilon)(1 - \varepsilon^2)} \qquad \ldots\ldots\ldots (7)$$

2 (ii) The distribution at the second focus from an elementary annular element

The image ellipse having semi-major and minor axes A and B is uniformly irradiated. If the annular ring is a radiator the image of the sphere in the ring is circular and is formed by rotating the ellipse about its centre.

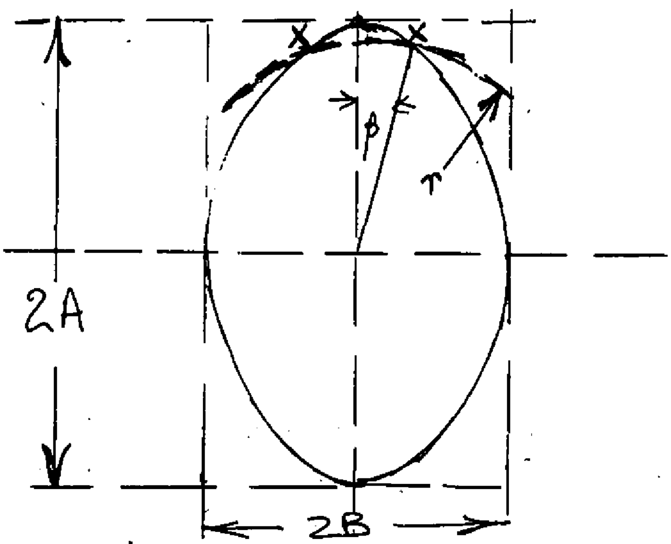

Fig. 2
Image in vertical plane at second focus of a sphere reflected by elementary area at P

Rotating the ellipse through a complete circle gives uniform intensity in the circular area $r < B$. For unit intensity within this circle the intensity in $A > r > B$ is

$$i' = \frac{2}{\Pi} \beta \qquad \ldots\ldots\ldots\ldots\ldots (8)$$

where $\sin \beta$ is obtained from the equation of the ellipse and the co-ordinates of X (see Fig. 2)

$$\frac{r^2 \cos^2 \beta}{A^2} + \frac{r^2 \sin^2 \beta}{B^2} = 1$$

i.e. $$\sin \beta = \left\{\frac{A^2}{r^2} - 1\right\}^{\frac{1}{2}} \cot \Psi \qquad \ldots\ldots\ldots\ldots\ldots (9)$$

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For $r > A$ the intensity is of course zero. From equation (5) we can obtain $\cot \Psi$ as

$$\cot \Psi = \frac{2\varepsilon + (1 + \varepsilon^2) \cos \theta}{(1 - \varepsilon^2) \sin \theta} \qquad \text{...............} (10)$$

2 (iii) <u>The integrated intensity for the whole mirror surface</u>

In Fig. 1 the element of energy incident at P is

$$\delta^2 E = I R^2 \sin \theta \, d\theta \, d\phi \qquad \text{...............} (11)$$

where θ and ϕ are angular spherical co-ordinates

and I is the radiant intensity at the surface of the sphere.

At $^{\Sigma}2$, $\delta^2 E$ is distributed over the area Π AB, giving an intensity from (6), (7) and (11).

$$\delta^2 i = \frac{\delta^2 E}{\Pi AB} = \frac{I \sin \theta \, d\theta \, d\phi (1 - \varepsilon^2)^2 [(1 + \varepsilon^2) \cos \theta + 2\varepsilon]}{(1 + \varepsilon^2 + 2\varepsilon \cos \theta)^3} \qquad \text{....} (12)$$

Hence the intensity on integrating for the annulus, i.e. $0 < \phi < 2\Pi$ is from equations (8) and (12)

$$\int_i' \delta^2 i = \delta i = \frac{2 I (1 - \varepsilon^2)^2 ((1 + \varepsilon^2) \cos \theta + 2\varepsilon) \sin \theta}{(1 + \varepsilon^2 + 2\varepsilon \cos \theta)^3} \times \qquad \text{...............} (13)$$

$$\times \frac{2}{\Pi} [\sin^{-1} \cot \Psi \left\{ \left(\frac{A}{r}\right)^2 - 1 \right\}^{\frac{1}{2}}] d\theta$$

where for $r > A$ the $\sin^{-1}$ term is replaced by zero

and where $r < B$ by $\frac{\Pi}{2}$. Now, because of the discontinuity at $r = A$ or B we need to find the relation between r and θ so that

$$r = B \text{ at } \theta_1$$
$$r = A \text{ at } \theta_2$$

i.e. rearranging equation (6), $\cos \theta_1 = \dfrac{(1 - \varepsilon^2) \frac{r}{R} - (1 + \varepsilon^2)}{2\varepsilon}$ .... (14)

and $\cos \theta_2$ is given by equation (7) with A equal to r, i.e.

$$\frac{r}{R} = \frac{(1 + 2\varepsilon \cos \theta_2 + \varepsilon^2)^2}{(1 - \varepsilon^2)(2\varepsilon + (1 + \varepsilon^2) \cos \theta_2)} \qquad \text{...............} (15)$$

Writing the term in the brackets [ ] in equation (13) as $f(r,\theta)$ the values of f for the integration over $0 < \theta < \theta_{max}$ are as given by equations (14) and (15) and are shown in Fig. 3. Writing equation (13) as $\delta i = f(r,\theta) F(\theta) d\theta$

we have $i = \int_0^{\theta_{max}} f(r,\theta) F(\theta) d\theta$

where θ_{max} is obtained from the diameter of the mirror.

If $\theta_1 < \theta_{max}$

$$i = \int_0^{\theta_1} F(\theta) d\theta + \frac{2}{\Pi} \int_{\theta_1}^{\theta max} F(\theta) \beta \, d\theta \qquad \text{...............} (16)$$

If θ_2 is less than θmax then the second integral proceeds only so far as θ_2 where β is zero

and for $\theta_1 > \theta_{max}$

i.e. from (14) $\dfrac{r}{R} < \dfrac{1 + \varepsilon^2 + 2\varepsilon \cos \theta \max}{(1 - \varepsilon^2)}$

and $i = \int_{\theta}^{\theta max} \frac{2\,I\,(1 - \varepsilon^2)^2\,((1 + \varepsilon^2)\cos\theta + 2\varepsilon)\sin\theta\, d\theta}{(1 + \varepsilon^2 + 2\varepsilon\cos\theta)^3}$

$= \frac{I\,(1 - \varepsilon^2)\sin^2\theta_{max}}{(1 + \varepsilon^2 + 2\varepsilon\cos\theta_{max})}$ ....(17)

There can be no intensity at all outside A_{max}

i.e. $i = 0$ when $\frac{r}{R} > \frac{1 + \varepsilon}{1 - \varepsilon}$ ....(18)

Equations (13) - (16) determine the non-uniform part of the distribution and equation (17) the uniform part.

3. A disc source

The above analysis has been for a sphere but it is possible to repeat it for a disc source, the essential difference being that distances in the vertical plane through F_2 perpendicular to F_1 and F_2 are reduced by $\cos\theta$. This is because the projection of the disc perpendicular to PF, is $\cos\theta$ of that for the sphere. With the same notation as for the spherical source we have for a disc normal to the axis of the ellipse, the image ellipse equivalent to Fig. 2 given by Fig. 4.

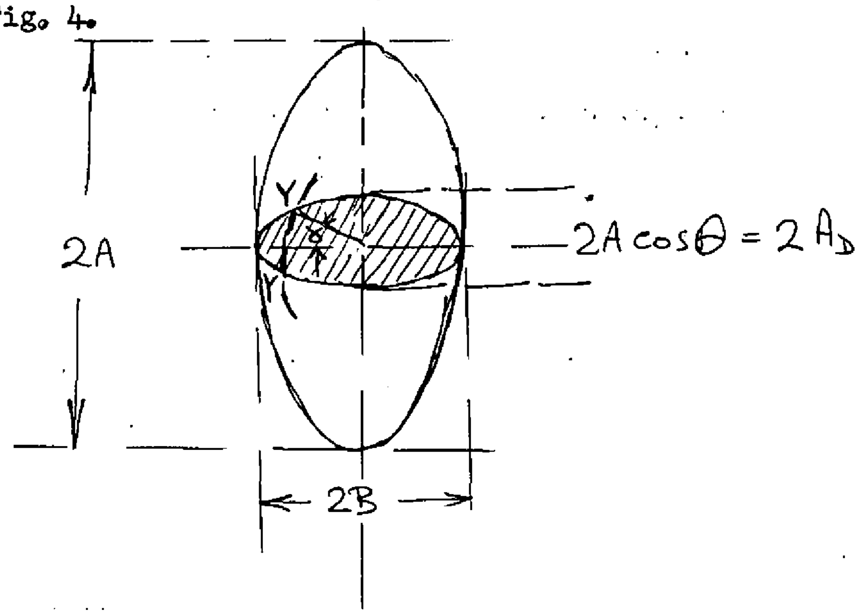

Fig. 4
Image in vertical flame at second form due to a disc reflected by elementary area P

2 B is now the major axis since for the other axis $2\,A_D$ equal to $\frac{2\,B\cos\theta}{\cos\Psi}$ is always less than 2 B.

Since both the area of the image ellipse and the element of energy are both reduced by $\cos\theta$, $\delta^2 i$ is unaltered. The integrated intensity is thus given by equation (17) provided r/R is less than the minimum value of $\frac{B\cos\theta}{\cos\Psi}$ and the intensity is zero if r/R is greater than $1 + \varepsilon/1 - \varepsilon$. The disc image is therefore of the same intensity as that for the sphere but the uniform zone is in the circle r/R less than the minimum value of $A\cos\theta$ i.e. 1.95 instead of $r/R < 3.8$.

We have

$A_D = A\cos\theta$

The demarkations of the integration zones are

$r = B$ as before at θ equal to θ_1

and $r = A_D$ at θ equal to θ_3

so that we have equation (14) as before and

$$\frac{r}{R} = \frac{(1 + 2\varepsilon \cos\theta_3 + \varepsilon^2)^2}{(1-\varepsilon^2)(2\varepsilon + (1+\varepsilon^2)\cos\theta_3)}$$

instead of equation (15). The relation between r/R and $\cos\theta_3$ is shown in Fig. 3. We require the angle γ (see Fig. 5) which as before we determine by solving the equation to the ellipse to obtain the point Y.

i.e. $$\frac{r^2}{A_D^2}\sin^2\gamma + \frac{r^2\cos^2\gamma}{B^2} = 1$$

Hence $$\sin^2\gamma = \sqrt{\left\{\frac{B^2}{r^2} - 1\right\} \Big/ \left\{\frac{\cos^2\psi}{\cos^2\theta} - 1\right\}} \qquad \text{.............. (19)}$$

Hence the function 'f' for the disc is

$$f_D = \frac{2}{\pi}\cdot\gamma$$

with γ given by equation (19). The integration follows that for the sphere except that θ_3 replaces θ_1 and θ_1 replaces θ_2 as limits of integration and γ replaces β in equation (16).

4. Application to a carbon arc source

For the carbon arc source in question[1] the focal lengths are 11 in. and 55 in. so that ε is $2/3$ and Fig. 3 has been drawn for this value of ε. Fig. 3 shows that the region where f is between zero and unity is a relatively small one and can be reasonably neglected. The diameter of the mirror is 24 in. so that the value of θ_{max} for the carbon arc source is 60° 27'.

For this 'intermediate' area, bounded in Fig. 3 by the lines $\beta = 0$ and $\beta = 1$ and the r/R axis the limiting values of r/R are $\frac{1+\varepsilon}{1-\varepsilon}$ and $\frac{1+\varepsilon^2+2\varepsilon\cos\theta_{max}}{1-\varepsilon^2}$

i.e. for ε equal to $2/3$ and $\cos\theta_{max}$ equal to 0.49

the intermediate area is $5.0 > \frac{r}{R} > 3.8$

$$\text{and } \cos\theta_1 = \frac{r/R - 2.6}{2.4} \qquad \text{.............. (20)}$$

We can then write from equation (16) for this region

$$i = 2I\int_0^{\cos^{-1}\frac{\frac{r}{R}-2.6}{2.4}} \frac{\frac{25}{81}\left(\frac{13}{9}\cos\theta + \frac{4}{3}\right)\sin\theta\, d\theta}{\left(\frac{13}{9} + \frac{4}{3}\cos\theta\right)^3} \qquad \text{.............. (21)}$$

For $\frac{r}{R} < 3.8$ we have from equation (21) with the upper limit as 60° 27'

$$i = 0.052\ I$$

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This and the result of computing equation (20) are shown in Fig. 4 for a particular carbon arc source[1] corresponding to a black body of 4000°K, i.e. I is 325 cal.cm^{-2}sec^{-1} so that i is 16.7 cal.cm^{-2}sec^{-1} which is about 26 per cent greater than the measured maximum intensity of carbon arc source. The distribution for a disc source has similarly been computed but no simplification can be made for the larger intermediate area.

5. Discussion

The distribution of radiation has been calculated for a sphere and a disc source at the focus of the J.F.R.O. mirror and the results are given in Fig. 5. They give a reasonable value for the maximum intensity. The carbons in the apparatus are 11/16 in. diameter so R is 11/32 in. The image of a spherical source should therefore be of uniform intensity within a radius of 3.8 x 11/32 in., i.e. $1\frac{1}{4}$ in. and for a disc source within $\frac{5}{8}$ in. radius. In fact, the actual distribution varies continuously and there is no region of uniform intensity. In Fig. 6 the actual distribution is horizontal and vertical direction is compared with that calculated for a disc. In the apparatus there is considerable obscuration by the arc holder and the anode rod and this distorts the image, and reduces the maximum intensity. From the analysis above, it appears that the effect of obscuration is primarily to diminish the size of the source, the effect on the maximum intensity not being unduly large.

6. Reference

(1) Hinkley, P.L. "A source of high intensity radiation employing an arc lamp and an ellipsoidal mirror." F.R. Note 270/1956.

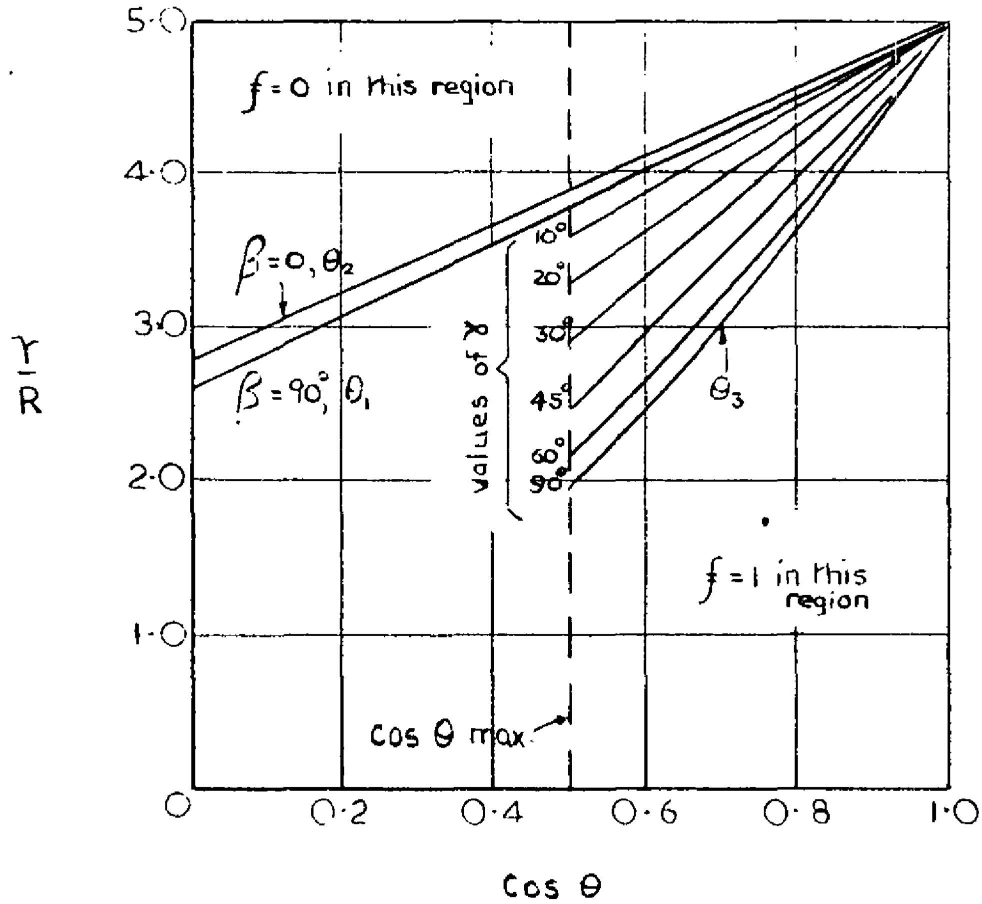

FIG. 3. DIAGRAM FOR DETERMINING INTEGRATION LIMITS (SPHERE & DISC) FOR $\epsilon = 2/3$

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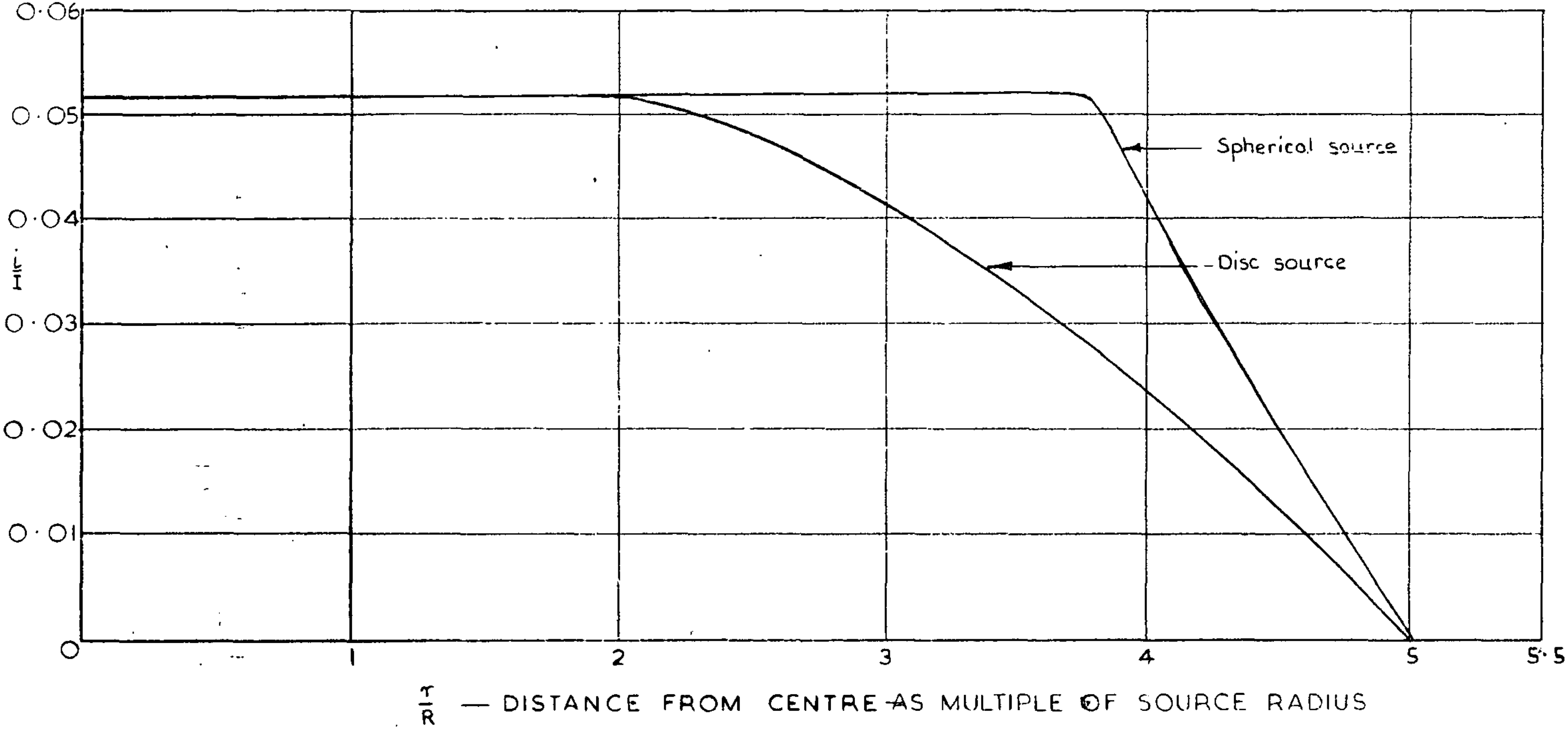

FIG. 5. CALCULATED DISTRIBUTION OF RADIATION AT IMAGES OF DISC AND SPHERICAL SOURCES

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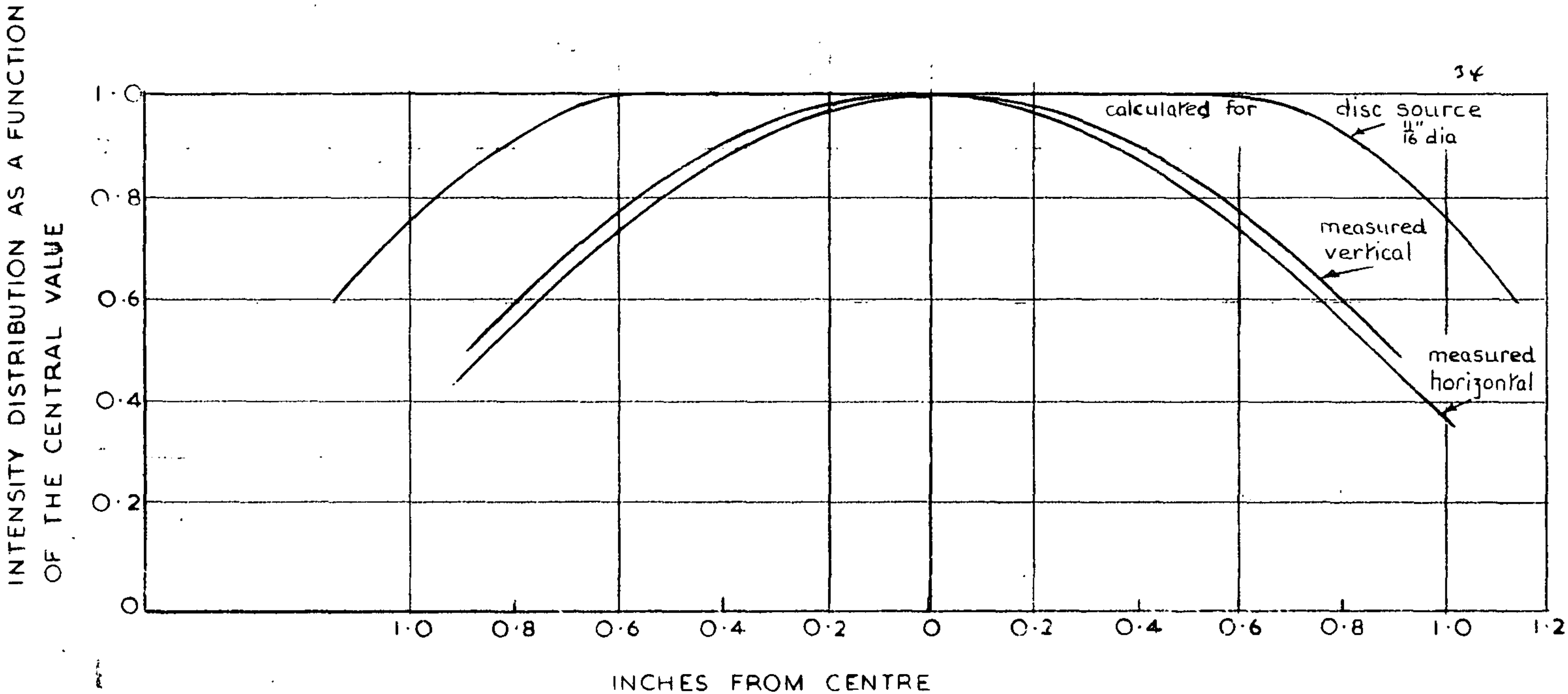

FIG. 6. DISTRIBUTION OF RADIATION AT SECOND FOCUS CALCULATED AND MEASURED — NORMALISED TO BE EQUAL AT THE CENTRE