F.R. Note No. 94/1954
Research Programme
Objective E/3/7(P)

DEPARTMENT OF SCIENTIFIC AND INDUSTRIAL RESEARCH AND FIRE OFFICES' COMMITTEE
JOINT FIRE RESEARCH ORGANIZATION

THE SCALING OF DIMENSIONS IN HEAT CONDUCTION PROBLEMS

by

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Summary

This note investigates the relation between the growths of temperature in similar structures, of different sizes, heated under comparable conditions. It shows that the dimensions in which there is no component of heat flow play no part in the thermal problem and that, in the direction of heat flow, the times to attain any specified temperature at corresponding points vary as the square of the dimensional scales of the structures. The square law relation is shown to apply where the thermal properties of the component materials vary with temperature and where, with one exception, the effects of contained moisture are considered. Various other effects are also examined.

The use of the B.S.476 time-temperature furnace curve in the conduct of fire-resistance tests precludes the application of the scaling relation to similar structures, since it requires time to be a scaled quantity. An empirical modification of the relation to avoid this difficulty, and the extension of the relation to similar structures of materials with different thermal constants, are described in two further notes.

January, 1954.

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Introduction

This note investigates the relation between the growths of temperature in similar structures of different sizes, heated under comparable conditions. The relation permits the result of a time-temperature problem to be predicted for a given structure, provided the result of a similar problem on a similar structure of another scale is available. The application of the relation requires the use of a time scale dependent upon the scale of the structure, in order that the basic heat transfer equations may be satisfied. Where a prescribed time-temperature function (other than a step function) is applied to the boundary of the structure the relation cannot be used.

The relation applies to similar structures of different scales having the same component materials. It is not invalidated where some thermal processes other than pure conduction are involved, for example, the evaporation of contained moisture and the convective-radiative transfer at interfaces where there is imperfect thermal contact.

In two later notes, the relation is extended to include the use of the time-temperature furnace curve prescribed in B.S.476 for the conduct of fire-resistance tests and the use of materials of different thermal properties in similar structures.

Scaling of dimensions in a pure heat conduction problem

In a large number of heat conduction problems the solution required is in terms of the time that must elapse before a specified temperature is attained at a particular point under certain conditions, and in this note such times are related to the dimensional scale of the structure involved. The scale of temperature is essentially considered to be invariant.

The relation is derived by applying scaling factors to several of the quantities in the differential equations governing the flow of heat by conduction so that the equations remain satisfied. It is shown, in Appendices 1, 2 and 3, that this is so if time is scaled as the square of the scaling factor n applied to linear dimensions in which there is a component of heat flow. Other quantities, including heat flux per unit area, also need to be scaled, but this does not affect the solution of problems in which boundary conditions and results are in terms of temperature.

Dimensions in which there is no component of heat flow do not enter into the thermal problem and it is shown that they may be scaled by any convenient factors.

The following is an example of how the square law could be applied in solving a problem in which the flow of heat in a structure is by conduction only. Suppose the first structure considered be a 10 inch diameter column, at least five feet high, and that its exterior temperature is raised to and maintained at 1,000°C. Suppose that after $\frac{3}{4}$ hour the temperature attained 3 inches from the surface is 400°C. Then it can be predicted that the temperature of 400°C would be attained at a depth of 6 inches from the heated face of a similar structure, of diameter 20 inches, subjected to the same conditions, in a time of $2^2 \times \frac{3}{4} = 3$ hours.

It should be noted that if boundary conditions are dependent on time then in scaled versions of a problem, they must be dependent on scaled time. The application of this statement can be illustrated by considering a slightly modified boundary condition in the example already quoted. Suppose that the temperature of the heated face had been raised to 1000°C at the beginning of the experiment and then to 1,500°C after half-an-hour. Then the prediction concerning the 20 inch column would have applied to the boundary condition that the surface temperature were raised to 1000°C at the beginning of the experiment and to 1,500°C after a time 2^2 x half-an hour = 2 hours had elapsed.

Variations of thermal properties with temperature

The properties of a material affecting the process of conduction, i.e. thermal conductivity (K), density (ρ) and specific heat (S), generally vary with temperature. This, however, would not affect the validity of time-temperature predictions made on the basis of the square law relation. The proof of this fact is given in Appendix 4.

Phenomena associated with heat conduction in practice

Most practical problems, described broadly as "thermal conductivity problems", involve more processes than that of pure heat conduction. The following is a list of the more important associated processes which might be involved, together with statements as to their influence on the validity of the square law relation.

(a) Presence of water

If water is present in a structure it affects transient temperature distributions by absorbing heat as its temperature is raised and as it is converted to steam. It is shown in Appendix 5 that these two effects would not influence scaling. The fact that water vapour occupies a greater volume than the same weight of water is also compatible with scaling.

It is found, in fire-resistance tests, that water vapour migrates from one part of a structure to another. Insofar as this is a diffusion process, it should again scale according to the square law relation.

(b) Cavities and imperfect thermal contact at interfaces

In composite structures the thermal contact at interfaces is never perfect and the heat transfer across the surfaces involves such processes as radiation. In general, however, imperfect thermal contact only amounts to the inclusion of a small thermal resistance in a structure which already has a high thermal resistance and its effect on scaling may, therefore, be neglected.

Imperfect thermal contact is the limiting case of the presence of cavities in a structure, and where the heat flux through the cavities is comparable with the heat flux by conduction, their effect must be taken into account. In any cavity heat transfer will be by radiation and convection as well as by conduction. In Appendices 1, 2 and 3 it is shown that heat flux per unit area is a scaled quantity since it depends on temperature gradient ($\partial\theta/\partial x$) and x has been scaled. Radiative heat transfer, on the other hand, is dependent on temperature difference and configuration factor and it would, in practice, be impossible to scale it in the manner required. The same may be said of convective heat transfer.

Where cavities exist in a structure, therefore, their effect on scaling may only be neglected if, from their geometry relative to the structure, it can be seen that they do not play a substantial part in heat transfer.

(c) Cooling to the atmosphere

As heat flux per unit area is a scaled quantity, the existence in a structure of cooling to the atmosphere is incompatible with scaling.

Cooling frequently plays a vital part in the thermal behaviour of many structures and it would be valuable if steps could be taken to allow scaling in these circumstances.

Two such steps are possible. Firstly the scaling relation can be experimentally amended. The derivation of such an amendment is to be described in a later note where it is suggested that the power law relating time and a dimension should be a little greater than two.

Secondly, the surface of a structure to be tested could be treated so that its rate of cooling is altered to represent that from the structure for which results are to be predicted. The results would not then refer to the original unaltered structure. This is a serious limitation of the technique.

Conclusions

If the times to attain specified temperatures at various points in a structure being heated by conduction are known, then the times to attain the same temperatures in scale models of the structure may be predicted, time being considered to scale as the square of linear dimensions.

Processes other than pure heat conduction are involved in many practical problems such as those of fire-resistance. The more important ones have been considered and it has been shown that most of them are compatible with the scaling described.

The relation can be applied to many fire-resistance problems. Two serious limitations exist however. Firstly, since time is a scaled quantity, the furnace temperature must follow a law based on scaled time. Secondly, all wall and floor fire-resistance problems involve cooling to the atmosphere, which does not fit in easily with the scaling relation described. In a later note the scaling relation will be experimentally amended to allow predictions in these circumstances.

Appendix 1 Scaling of three dimensional flow of heat by conduction

The flow of heat by conduction is governed by the following equations:-

$$\begin{cases} F_x = -K.\delta y.\delta z\dfrac{\partial\theta}{\partial x} & \text{--- (1a)} \\ F_y = -K.\delta x.\delta z\dfrac{\partial\theta}{\partial y} & \text{--- (1b)} \\ F_z = -K.\delta x.\delta y\dfrac{\partial\theta}{\partial z} & \text{--- (1c)} \end{cases}$$

and

$$\frac{\partial F_x}{\partial x}\delta x + \frac{\partial F_y}{\partial y}\delta y + \frac{\partial F_z}{\partial z}\delta z = -\rho s.\delta x.\delta y.\delta z\frac{\partial\theta}{\partial t} \quad \text{--- (2)}$$

where the symbols are defined in Appendix 6

If x, y and z be scaled by the factor n whilst θ, K, ρ and s be left unscaled, then the equations will remain consistent if modified as follows:-

$$\begin{cases} nF_x = -K\,\delta ny\,.\,\delta nz\,\dfrac{\partial\theta}{\partial nx} & \text{------ (3a)} \\ nF_y = -K\,\delta nx\,.\,\delta nz\,\dfrac{\partial\theta}{\partial ny} & \text{------ (3b)} \\ nF_z = -K\,\delta nx\,.\,\delta ny\,\dfrac{\partial\theta}{\partial nz} & \text{------ (3c)} \end{cases}$$

and

$$\frac{\partial nF_x}{\partial nx}\delta nx + \frac{\partial nF_y}{\partial ny}\delta ny + \frac{\partial nF_z}{\partial nz}\delta nz = -\rho s\,.\,\delta nx\,.\,\delta ny\,.\,\delta nz\,.\,\frac{\partial\theta}{\partial n^2 t} \quad \text{-- (4)}$$

Thus since x, y and z have become nx, ny and nz then t becomes n^2t and rate of flow of heat per unit area $F_x/\delta y.\delta z$ or $F_y/\delta x.\delta z$ or $F_z/\delta x.\delta y$ becomes $\frac{1}{n}(F_x/\delta y.\delta z)$ or $\frac{1}{n}(F_y/\delta x.\delta z)$ or $\frac{1}{n}(F_z/\delta x.\delta y)$ (at corresponding times).

The quantity of heat present per unit volume (at corresponding times) is unaltered and thus the total quantity of heat present scales as the volume i.e. as n^3.

The total heat flux scales as the product (area x rate of flow of heat per unit area) i.e. as $n^2 \times \frac{1}{n} = n$.

Appendix 2. Scaling of two dimensional flow of heat by conduction

If there is no component of heat flow in the z dimension, then the flow of heat by conduction is governed by the following equations:-

$$\begin{cases} F_x = -K\,\delta y\,.\,\delta z\,\dfrac{\partial\theta}{\partial x} & \text{-------- (5a)} \\ F_y = -K\,\delta x\,.\,\delta z\,\dfrac{\partial\theta}{\partial y} & \text{-------- (5b)} \end{cases}$$

and

$$\frac{\partial F_x}{\partial x}\delta x + \frac{\partial F_y}{\partial y}\delta y = -\rho s\,.\,\delta x\,.\,\delta y\,.\,\delta z\,.\,\frac{\partial\theta}{\partial t} \quad \text{------ (6)}$$

where the symbols are defined in Appendix 6.

If x and y be scaled by n and z by m, whilst θ, K, ρ and s be left unscaled, then the equations will remain consistent if modified as follows:-

$$\begin{cases} mF_x = -K\,\delta ny.\,\delta mz\,\dfrac{\partial\theta}{\partial nx} & \text{------------}(7a) \\ mF_y = -K\,\delta nx.\,\delta mz\,\dfrac{\partial\theta}{\partial ny} & \text{------------}(7b) \end{cases}$$

and

$$\frac{\partial mF_x}{\partial nx}\,\delta nx + \frac{\partial mF_y}{\partial ny}\,\delta ny = -\rho s\,\delta nx.\,\delta ny.\,\delta mz\,\frac{\partial\theta}{\partial n^2 t} \quad \text{----}(8)$$

Thus, since x and y have become nx and ny, then t becomes n^2t and rate of flow of heat per unit area $F_x/\delta y\delta z$ or $F_y/\delta x\delta z$ becomes $\frac{1}{n}(F_x/\delta y\delta z)$ or $\frac{1}{n}(F_y/\delta x\delta z)$ (at corresponding times).

The quantity of heat present per unit volume (at corresponding times) is unaltered and thus the total quantity of heat present scales as the volume i.e. as n^2m.

The total heat flux scales as the product (area x rate of flow of heat per unit area) i.e. as $nm \times \frac{1}{n} = m$.

Appendix 3. Scaling of one dimensional flow of heat by conduction

If the heat flow is unidirectional and is along the x dimension then the equations governing the flow of heat by conduction are:-

$$F_x = -K\,\delta y.\,\delta z\,\frac{\partial\theta}{\partial x} \quad \text{------------}(9)$$

and

$$\frac{\partial F_x}{\partial x}\,\delta x = -\rho s\,\delta x.\,\delta y\,\delta z.\,\frac{\partial\theta}{\partial t} \quad \text{------------}(10)$$

where the symbols are defined in Appendix 6.

If x be scaled n, y by g and z by m, whilst θ, K, ρ and s be left unscaled, then the equations will remain consistent if modified as follows:-

$$\frac{gm}{n}F_x = -K\,\delta gy.\,\delta mz.\,\frac{\partial\theta}{\partial nx} \quad \text{------------}(11)$$

and

$$\frac{\partial \frac{gm}{n}F_x}{\partial nx}\,\delta x = -\rho s\,\delta nx.\,\delta gy.\,\delta mz\,\frac{\partial\theta}{\partial n^2 t} \quad \text{-----}(12)$$

Thus since x has become nx, then t becomes n^2t and rate of flow of heat per unit area $F_x/\delta y\delta z$ becomes $\frac{1}{n}(F_x/\delta y\delta z)$ (at corresponding times).

The quantity of heat present per unit volume (at corresponding times) remains unaltered and thus the total quantity of heat present scales as the volume i.e. as n g m.

The total heat flux scales as the product (area x rate of flow of heat per unit area) i.e. as $\frac{mg}{n}$.

Appendix 4. Variation of thermal properties with temperature

If, in Appendices 1, 2 or 3, thermal conductivity, density and specific heat were functions of temperature, then in place of K, ρ and s it would be necessary to write $Kf(\theta)$, $\rho f'(\theta)$ and $sf''(\theta)$. Since the quantity θ is unscaled, these factors would appear unaltered in the modified equations and the consistency of the equations would not be affected. Thus variation of these quantities with temperature would not be incompatible with the scaling as described.

Appendix 5. Presence of Water

The principal effect of the presence of water in a structure is that it absorbs heat as its temperature is raised and as it is converted into steam. These two modes of behaviour are described by the following two equations:-

$$\frac{\partial F_x}{\partial x}\delta x + \frac{\partial F_y}{\partial y}\delta y + \frac{\partial F_z}{\partial z}\delta z = -\rho' \delta x . \delta y . \delta z \frac{\partial \theta}{\partial t} \quad ----(13)$$

and

$$\frac{\partial F_x}{\partial x}\delta x + \frac{\partial F_y}{\partial y}\delta y + \frac{\partial F_z}{\partial z}\delta z = -l . \delta x . \delta y . \delta z \frac{\partial \rho'}{\partial t} \quad ---- (14)$$

where l is the latent heat of steam

ρ' is the mass of water present per unit volume

and the remaining symbols are defined in Appendix 6.

Equation (13) is of precisely the same form as one of the conduction of heat equations and equation (14) is similar in that l and ρ' are not subject to scaling just as θ, s and ρ are not in equation (13).

It, therefore, follows automatically that the equations will remain consistent when scaled and that so far as the effect of absorbing heat is concerned the presence of water in a specimen will not prejudice scaling.

Appendix 6. List of symbols

x, y and z	Cartesian co-ordinates
t	Time
Θ	Temperature rise
F_x, F_y and F_z	Components of heat flux, through defined areas, in the directions x, y and z
K	Thermal conductivity
ρ	Density
s	Specific heat
l	Latent heat of steam
ρ'	Mass of water present in a structure per unit volume
n	Scaling factor applied to dimensions in which there is a component of heat flow
m, g	Scaling factors applied to dimensions in which there is no component of heat flow

KC.