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EXTREME VALUE THEORY AND FIRE RESISTANCE

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SUMMARY

Safety measures in a building based on average fire severity and average fire resistance are likely to be less efficient than measures based on extreme values of these two factors. What is ideally required is a minimum fire resistance to cope with the maximum fire severity likely to be reached in a building. The costs and benefits involved in such a degree of safety can be evaluated later.

This Note, however, is concerned only with the fire resistance of a structural element. The statistical properties of the minimum fire resistance could be studied with the aid of data from a small number of tests. Such an analysis would require the application of the theory of extreme values as illustrated in the example.

KEY WORDS: Fire resistance, structural elements, statistical method.

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INTRODUCTION

Most problems of design of structures involve the consideration of extreme values: high values in the evaluation of loads and low values in the evaluation of strengths. However, many engineers, scientists and technologists are not familiar with the statistical concepts of extremes, but only with classical concepts related to central tendencies like average, mode, dispersion etc which in this type of analysis, although important, are not predominant. The use of safety factors applied to the average of loads or of strengths instead of to their extreme values can be misleading. These views were expressed in a seminar on "Engineering applications of statistical extremes" held by Nato Advanced Study Institute at Faro in Portugal in 1967.

Fire research is concerned with the safety of lives and materials from fire. Hence, as expressed in the Nato seminar, safety measures in a building based on average fire severity and average fire resistance are likely to be less efficient than measures based on extreme values of these two factors. What is required is a minimum fire resistance to cope with the maximum fire severity that would be likely in a building. Then the economist has to measure the expectation (or any other suitable function) of gain or utility for the acceptable degree of safety. Such economic analyses can be taken up only after the statistical concepts and evaluation are elucidated. This Note is concerned with the application of extreme value theory for studying the statistical properties of the minimum fire resistance of a structural element.

FIRE RESISTANCE

A structural element used in the construction of a building may be required to have a certain minimum fire resistance and this minimum requirement is often prescribed in the building regulations. The adequacy and economic justification of this minimum needs some verification though this is not pursued here. The ability of a structural element to meet the required standard is usually judged in a single standard test. Statistically this is not satisfactory since repeated trials are necessary to allow for the chance variation.

As in the case of research experiments, it would be ideal to carry out two or more tests (within economic limits) and measure the total period of satisfactory performance (ie time to failure) in each test instead of terminating the test when the required level of fire resistance is reached. Thereafter the results of the tests could be analysed. The classical approach would be to accept or reject the specimen tested on the basis of the average or median fire resistance. The probability that the performance under test conditions would be less than, say, the estimated average is about 50 per cent. The precise level of probability, of course, depends upon the stochastic process governing the time to failure of the item tested. It is questionable whether it is wise to adopt the 50 per cent level of probability for safety measures.

A certain risk of damage to life and property is associated with each level of probability. Determination of an optimum level requires the application of decision theory under uncertainty and this could be studied later. However, the safest approach would be to base decisions on the expected minimum fire resistance of the element taking into account the variance of the test results. The probability that the actual fire resistance would be less than this expected minimum is much smaller than the corresponding probability for the expected average. If this minimum is not significantly different from the requirement as specified in building regulation the element may be regarded as quite safe for use in the type of building considered. The requirement is for a minimum fire resistance and not for an average. It is but reasonable to compare like with like.

For economic reasons only a few tests can be conducted for a given construction and hence the observed minimum is usually obtained from a small sample. If the set of tests were repeated a number of times it is to be expected that different minima could be observed in each replication. What is required is the average value of the minima in such repeated tests without actually carrying out these replicated trials. The solution to this problem is given by the statistical theory of extreme values concerned with small samples from a known distribution. This is slightly different from the asymptotic theory of extreme values. The use of the asymptotic theory is also explained for obtaining the minimum in large samples.

MINIMUM FIRE RESISTANCE - SMALL SAMPLES

Consider the variable X which is the fire resistance of a structural element under a design load prescribed in the test. The variable has a probability distribution the exact nature of which could be established only by a detailed study of the stochastic process governing the spread of fire. However, it is well known that a variable measured in units of time is likely to have an exponential or a logarithmic normal probability distribution. These two distributions take care to some extent of the skewness which normally characterises a time variable.

For this study, however, the logarithmic normal will be used for purposes of illustration. This distribution was assumed in a statistical analysis of the fire resistance of laminated timber columns.[1] Table 1 shows the observed values of fire resistance of the columns converted into logarithmic units.

TABLE 1

Fire resistance (log minutes)

Glue \ Species	Douglas fir	Western Hemlock	Redwood	Western Red Cedar
Urea	1.7364	1.8633	1.6721	1.5378
Casein	1.7243	1.3617	1.7324	1.7889
Resorcinal	1.8707	1.6902	1.6484	1.6365
Phenolic	1.6532	1.8357	1.8779	1.5911
Average	1.7462	1.6877	1.7327	1.6386

The averages given in the last row of the table are the location parameters of the fire resistance times for the different species. The fire resistance times are assumed to be independently and identically distributed random variables with a common variance. This common variance given by the residual variation in the analysis was 0.0041. It is better to use the residual error since it is free from the effects of the three factors glue, load and shape apart from species.

Consider now the columns made of Douglas fir. They had an average fire resistance of 1.7462 (μ) with a standard error of 0.0640 (σ) which is the square root of the common variance. It is known that, since $x' = \log x$ is normally distributed, the variable

$$t = \frac{x' - \mu}{\sigma} \qquad \text{..... (1)}$$

has a standard normal distribution with mean zero and variance unity. Let x'_{1n} be the minimum in a sample of n tests. Then the corresponding minimum value of t is given by

$$t_{1n} = \frac{x'_{1n} - \mu}{\sigma} \qquad \text{..... (2)}$$

In the example considered $n = 4$. In repeated trials of 4 tests each trial, with the observations obeying a standard normal, the expected value of t_{1n} is -1.0294[2]. (It is not necessary here to discuss in detail the method by which this expectation has been calculated). From (2),

$$\begin{aligned} E(x'_{1n}) &= \mu - 1.0294\sigma \\ &= 1.7462 - 0.0659 \\ &= 1.6803 \qquad \text{..... (3)} \end{aligned}$$

Hence, based on the results of the tests, the expected value of the minimum fire resistance of laminated columns of Douglas fir with a variety of glues in a sample of four is 1.6803 ie 47.9 minutes. (The minimum observed in the test was 1.6532 min). The probability that the actual fire resistance is less than or equal to 1.6803 is 0.25.

The variance of μ is $\frac{\sigma^2}{4}$ and the variance of t_{1n} for $n = 4$ is 0.4917[3]. Hence the variance of x'_{1n} is given by

$$\begin{aligned} V(x'_{1n}) &= \sigma^2\left\{\frac{1}{4} + V(t_{1n})\right\} \\ &= \sigma^2\left\{\frac{1}{4} + 0.4917\right\} \\ &= 0.0030 \qquad \text{..... (4)} \end{aligned}$$

The skewness and kurtosis coefficients of the distribution of x'_{1n} indicate that this variable is non-normal. Hence it is difficult to construct the confidence limits for x'_{1n} but this problem might be attempted later.

The expected minimum fire resistance for columns of the other three species are as follows:

Western Hemlock	-	1.6218
Red Wood	-	1.6668
Western Red Cedar	-	1.5727

The variance for the three species is, of course, the same as the variance for Douglas fir viz 0.0030.

MINIMUM FIRE RESISTANCE - LARGE SAMPLES

According to extreme value theory, for large n, t_{1n} has the first asymptotic distribution of the smallest value true of parent distributions belonging to the exponential family. The logarithmic normal belongs to this family. Hence the asymptotic density function of t_{1n} is given by

$$\psi_1(t_1) = \alpha_1 e^{y_1 - e^{y_1}} dt_1 \qquad \text{..... (5)}$$

where

$$y_1 = \alpha_1 (t_1 - u_1) \qquad \text{..... (6)}$$

In (6), the parameter u_1, is known as the characteristic smallest value and the value of the intensity function of the par ent distribution at u_1. The parameters are solutions of the following equations

$$F(u_1) = \frac{1}{n} \quad \text{and} \qquad \text{..... (7)}$$

$$\alpha_1 = n \cdot f(u_1) \qquad \text{..... (8)}$$

$F(t)$ is the (cumulative) distribution function of t with $f(t)$ as the density function. The distribution function of t_1 corresponding to the density (5) is

$$\varphi_1(t_1) = e^{-e^{y_1}} \qquad \text{..... (9)}$$

Suppose it is required to find the minimum fire resistance such that there is a probability of, say, 0.01 that the actual fire resistance is less than or equal to it. Then, from (7), $n = 100$ and u_1 is the modal (most probable) value of such a minimum for the transformed variable t_1.

Since $F(t)$ has a standard normal distribution, from tables of this function,

$$u_1 = -2.33 \qquad \text{..... (10)}$$

Since $f(u_1) = 0.0264$, from (8),

$$\alpha_1 = 2.64$$

From (6) and (2)

$$t_1 = u_1 + \frac{y_1}{\alpha_1}$$
$$= \frac{x'_{1n} - \mu}{\sigma}$$

Hence

$$x'_{1n} = \mu + \sigma\left(u_1 + \frac{y_1}{\alpha_1}\right) \qquad \text{..... (11)}$$

For laminated columns of Douglas fir

$$x'_{1n} = 1.7462 + 0.064\left(-2.33 + \frac{y_1}{2.64}\right)$$
$$= 1.5971 + 0.0242\, y_1 \qquad \text{..... (12)}$$

The first term on the right hand side of (12) is the modal value of the minimum fire resistance in a sample of 100. The expected value of y_1 is -0.5772 which is the same as the expected value of the largest reduced value but with a negative sign.[4] The variance is $\frac{\pi^2}{6}$ or 1.6449. Hence

$$E(x'_{1n}) = 1.5831 \text{ and} \qquad \text{..... (13)}$$
$$V(x'_{1n}) = \frac{\sigma^2}{4} + (0.0242)^2\, 1.6449$$
$$= 0.00103 + 0.00096$$
$$= 0.00199 \qquad \text{..... (14)}$$

the standard error being 0.0446. From expression (13) the expected (average) value in a sample of 100 of the minimum fire resistance of Douglas fir columns is 1.5831. The actual fire resistance will be less than this value in only one out of 100 cases. The corresponding values for the other three species are

Western Hemlock	–	1.5246
Red Wood	–	1.5696
Western Red Cedar	–	1.4755

DISCUSSION

In problems concerned with safety the risk of the unfavourable event happening has to be reduced to a level acceptable to society. For this reason central tendencies like average, median, mode etc are not efficient criteria for judging the performance of safety measures. The probability of exceeding or falling short of a median or an average is as high as 50 per cent approximately. With a much lower level of probability, use of extremes instead of, say, averages would reduce the risk considerably. For these reasons it would be ideal to carry out 2 or more fire resistance tests with samples of the same structural element and measure the total period of satisfactory performance or time to failure in each test instead of stopping the test when the required fire resistance level is reached. Thereafter the minimum fire resistance shown in these tests should be used as the basis for accepting or rejecting the structural element.

Like the average, the minimum is a variable subjected to the laws of chance. By applying the theory of extreme values for small samples it is possible to study the stochastic behaviour of the minimum obtained in a small number of tests. This has been illustrated in this paper using data from tests carried out to find the fire resistance of laminated columns made from different species of timber. The use of the asymptotic theory of extreme values has also been explained in order to study the behaviour of the minimum in a hypothetically large sample. For a given level of probability the modal value of the minimum could be obtained directly from expression (1), with the aid of tables of normal distribution. But it is not possible to estimate the expected value or the random variation (variance) of the minimum in this way. Such an analysis is possible only by an application of the extreme value theory.

Analysis of variance for judging the significance of the differences between various factors in regard to the minimum is a problem yet to be solved. In the classical analysis only the average effects of the factors are studied. In the tests used as an example in this paper factors like the type of glue were investigated. The influences of these factors on the minimum fire resistance need to be examined.

In calculating factors of safety, it is usual to consider the maximum possible severity or load say S_{max} and the minimum resistance or strength, say R_{min}. In the test x has been used to denote fire resistance. Like fire resistance (R), fire severity (s) has a probability distribution and S_{max} is the largest or maximum value from this distribution. The next logical step in this area of research is to obtain the distribution of S and then construct the joint probability distribution of S_{max} and R_{min} from which the distribution of the safety margin $X_e = R_{min} - S_{max}$ could be obtained. Then, a safety index may be defined by

$$\beta_e = \frac{X_e}{\sigma_e} \qquad \text{..... (15)}$$

where σ_e is the standard error of X_e. A similar index has been suggested by Cornell[5] who has used R and S instead of their extreme values R_{min} and S_{max}. The confidence limits of β_e need to be calculated.

CONCLUSION

Minimum fire resistance is a better and safer yardstick than average fire resistance for evaluating the performance of structural elements in fire tests. By applying the statistical theory of extreme values it is possible to study the random behaviour of the minimum in small samples as well as large ones. The size of the sample depends upon the probability level chosen in conjunction with the risk one is prepared to tolerate.

In tests of laminated timber columns the expected values of the minimum fire resistance ranged from 1.5727 (37.4 minutes) for Western Red Cedar columns to 1.6803 (47.9 minutes) for Douglas fir. The probability that the actual fire resistance is less than the above mentioned minimum for each timber species is 0.25. The common standard error for these minima is 0.0548.

Extrapolating the test results it appears that the expected values of the minimum fire resistance in a sample of 100 tests would range from 1.4755 (29.9 minutes) for Western Red Cedar to 1.5831 (38.3 minutes) for Douglas Fir. The common standard error for these minima is 0.0446.

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