DEPARTMENT OF SCIENTIFIC AND INDUSTRIAL RESEARCH AND FIRE OFFICES' COMMITTEE
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THE SCALING OF HEAT CONDUCTION PROBLEMS

by

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Summary

This note derives the principles by which certain heat conduction
problems may be solved when the solution of a related problem is known.
The limitations in the application of these principles to the prediction
of the fire-resistance of building structures, tested in accordance with
B.S. 476 : 1953, are discussed.

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(1) Introduction

The solution of a heat conduction problem may be described mathematically as the application of boundary conditions to a pair of differential equations. The object of this note is to show that it is often possible to scale the various quantities in the equations in such a manner that whole series of problems reduce to one mathematical problem. When this problem is solved, possibly experimentally, the solutions of the others can be predicted directly.

It is assumed that temperature is the one quantity that must not be scaled (principally because the thermal properties of many materials vary with temperature) and in Appendices 1, 2 and 3 the relation between other scaling factors is derived, making this assumption. In applying the scaling principles to a series of problems, special attention must be paid to the boundary conditions to ensure that they still hold after the various quantities have been scaled. Many boundary conditions will, for example, concern heat flux per unit area and under these circumstances this quantity must not be subjected to scaling. Consideration of the equations given in the Appendices shows that the scale of heat flux per unit area may be considered invariant instead of temperature, but this case is not considered.

Very few practical problems involve only a process of heat conduction, and the effect on scaling of associated processes must be considered. In applying the scaling principles to the prediction of the fire-resistance of building structures (B.S. 476 : 1953) a further complication arises. A fixed furnace time-temperature curve has been prescribed in this Standard and in all fire-resistance tests the furnace temperature is controlled in accordance with this curve. Now with the method of scaling described, time is always, except in very special circumstances, a scaled quantity and thus any predictions based on a normal test result will apply to a furnace boundary condition referred to the wrong scale of time. The effect of this factor may, however, be considerably reduced by considering the time scaling factor to be raised to the power 0.8.

(2) The principles of scaling in a heat conduction problem

In Appendices 1, 2 and 3, the effect is considered of scaling dimensions in which there is a component of heat flow by a factor q, thermal conductivity by a factor m and the product density x specific heat (ρ s) by a factor n. Appendices 1 and 2 refer to cases in which there is no component of heat flow in one or two dimensions. Such dimensions do not enter into the thermal problem and may be scaled by any convenient factor. Total heat flux is the only quantity affected by this action, which therefore merely governs the power consumption from the heat source controlling the boundary conditions.

It is shown that the equations remain consistent and the problem thus unaltered provided time be considered to be scaled by the factor nq^2/m and heat flux per unit area by m/q.

The following is an example of the application of the above principles to a heat conduction problem.

Example 1

Suppose the first structure considered were a 10 inch diameter column, of considerable height, and that the exterior temperature were raised to and maintained at 1,000°C. Suppose that after ½ hour the temperature 3 inches from the surface rose to 400°C.

Then the time taken for the temperature to rise to 400°C at a point 6 inches from the surface of a 20 inch column could be predicted even though the structure might be composed of a material of three times the thermal conductivity and four times the thermal capacity (i.e. four times the product ρS).

The factors, $q = x'/x$, $m = k'/k$ and $n = \rho' S'/\rho S$

would have values of 2, 3 and 4 respectively, and thus the time scaling factor would have a value $nq^2/m = 4.2^2/3 = 16/3$. The temperature of 400°C would therefore be attained at a point 6 inches from the surface of the 20 inch diameter column in a time of $\frac{3}{4} \times 16/3 = 4$ hours.

It will be noted that in the above example the boundary condition does not involve scaled quantities (in particular time and heat flux per unit area). Examples including such conditions will be given in a later paragraph.

Most practical problems in which the heat flow is largely a process of heat conduction involve various other processes at the same time. The effect on scaling of the principal associated processes found in fire-resistance tests will now be discussed, together with the limitations in the application of the scaling technique described.

(3) Boundary conditions

In a practical problem boundary conditions will probably involve either time or heat flux per unit area (e.g. cooling to atmosphere) and any predictions concerning scaled structures would therefore apply for boundary conditions referring to scaled time or scaled heat flux per unit area. The following examples illustrate the above statement.

Example 2

Suppose, in Example 1, that the boundary condition applying to the first structure had been not merely that the external temperature had been raised to and maintained at 1,000°C but that after $\frac{1}{2}$-hour it had been raised a further 500°C. Then the predictions concerning the second structure would have referred to an initial external temperature of 1,000°C followed by a rise of 500°C after a time $\frac{1}{2} \times 16/3 = 2$ hours 40 minutes.

Example 3

Suppose the boundary condition in Example 1 had been that the exterior surface had been subjected to a heat flux per unit area of 1 cal/cm²/sec and it had been assumed that there were no losses from that surface. Now heat flux scales as m/q which, considering the second structure, has the value 3/2 = 1.5. The predictions concerning the second structure would therefore refer to a heat flux per unit area boundary condition at the exterior surface of 1.5 cal/cm²/sec.

So far as fire-resistance problems are concerned, the time boundary condition limitation is important since a fire-resistance furnace is made to follow a standard time-temperature curve. In a previous note, [1] solely concerned with the scaling of dimensions in a fire-resistance problem, an empirical relation has been derived to cater for the use of the standard time-temperature curve and an approximation of the relation is applicable to the more general method of scaling described in this note. The time scaling factor should be raised to the power 0.8 and thus becomes $(nq^2/m)^{0.8}$. A table of $a = b^{0.8}$ is given in Appendix 4.

In the same note [1] an empirical relation has been derived catering for the fact that cooling to the atmosphere cannot be scaled. This relation has not, however, been extended to cover scaling techniques in which K and the product ρS are scaled. Where cooling to the atmosphere is involved in fire-resistance problems (e.g. walls and floors) the scaling should therefore be planned such that m/q is a constant i.e. such that dimensions scale by the same factor as thermal conductivity.

(4) Variations of thermal properties with temperature

Where K and the product ρS are functions of temperature and should more properly be written $K f(\theta)$ and $(\rho S)f(\theta)$, then the new values of thermal conductivity and the product ρS must be of the form $m K f(\theta)$ and $n(\rho S)f'(\theta)$. The proof of this rule, in more specialised circumstances, is given in a previous note [2].

(5) The effect of the presence of water in a structure

The fact that two of the principal effects of water are compatible with the scaling of dimensions (i.e. use of the factor q) is proved in a previous note [2] and it is also stated that the third important effect, that of migration, is probably also compatible, to a close approximation.

The proof may easily be extended to cover the use of the factor m. The use of the scaling factor n, however, would imply that the equivalent weight of water per unit volume were also scaled by n.

(6) The effect of cavities and imperfect thermal contact at interfaces

In a previous note [2] it is stated that the effect, on scaling, of imperfect thermal contact at interfaces may be neglected, as also may the effect of cavities provided it can be seen, from their geometry, that they do not play a substantial part in heat transfer.

Where the cavities are substantial compared with the areas through which heat is flowing, the scaling laws given in this note will still apply, to a close approximation, provided heat flux per unit area remains unscaled, i.e. provided $q/m = 1$.

(7) The scaling of composite structures

Where a structure is composed of several different materials and heat flow is one-dimensional, the scaling factors applied to the different sections of the structure need not necessarily be the same, although strict limitations apply to the use of such a technique. The scales of heat flux per unit area and time must be constant throughout the material and thus nq^2/m and m/q must be constant. Solving between these equations gives the conditions that, throughout the structure, the product $q n$ and the quotient m/q must be constant.

The fact that the scaling factor q (applied to dimensions in which there is a component of heat flow) may be variable throughout a structure only has physical significance when the concept is confined to one dimension.

An interesting special case arises when $m n = 1$ i.e. when the products $K \rho S$ for two materials are the same and when $q/m = 1$. The two slabs of material, of thickness proportional to their thermal conductivities, will behave identically, irrespective of boundary conditions and whether or not they be in a composite structure.

(8) Conclusions

The principles of scaling described in this note may be applied to fire-resistance problems provided:-

1) if a standard time-temperature curve is used the time scaling factor be raised to the power 0·8,

2) if the structure be a wall or a floor, the ratio of the dimensional and thermal conductivity scaling factors (q/m) must be unity. This restriction also applies if large cavities exist in the structure.

   Alternatively if m (and also n) be unity the empirical scaling law given in a previous note may be used [1].

3) if water be present in a structure the quantity must bear a specified ratio to the thermal capacity of the containing material,

4) if thermal properties of component materials are functions of temperature then for the replacing materials the functions must be the same,

5) if composite structures are being considered and it is desired to vary m, n and q throughout the scaled version, the product q^n and the quotient q/m must be constant.

This note is entirely concerned with predicting times to attain specified temperatures. As mechanical failure in a fire-resistance test is generally a function of temperature the principles described are almost always directly applicable.

References

1. McGuire, J. H., "The scaling of dimensions in B.S. 476 fire resistance tests". Joint Fire Research Organization F.R. Note No. 95/1954.

2. McGuire, J. H., "The scaling of dimensions in heat conduction problems". Joint Fire Research Organization F.R. Note No. 94/1954.

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APPENDIX I

Scaling in a one dimensional heat conduction problem

The flow of heat by conduction in a one dimensional problem is governed by the following equations:-

$$F_x = -K.\delta_y.\delta_z \frac{\partial \theta}{\partial x} \qquad \ldots\ldots (1)$$

and

$$\frac{\partial F_x}{\partial x}.\delta x = -\rho S.\delta x.\delta_y.\delta_z \frac{\partial \theta}{\partial t} \qquad \ldots\ldots (2)$$

where the symbols are defined in Appendix 5.

If x, t, and the product ρS be scaled by the factors q, m, and n respectively whilst θ be left unscaled, then the equations will remain consistent if modified as follows:-

$$\frac{m}{q} F_x = -m K.\delta_y.\delta_z \frac{\partial \theta}{\partial q x} \qquad \ldots\ldots (3)$$

and

$$\frac{\partial \frac{m}{q} F_x}{\partial q x}.\delta q x = -n\rho S.\delta q x.\delta_y.\delta_z.\frac{\partial \theta}{\partial \frac{n q^2}{m} t} \qquad \ldots\ldots (4)$$

Thus time t becomes $\frac{n q^2}{m} t$ and the rate of flow of heat per unit area $F_x/\delta_y.\delta_z$ becomes $\frac{m}{q}\left(F_x/\delta_y.\delta_z\right)$

The dimensions y and z do not enter into the thermal problem and may be scaled by additional factors u and v. Such action would not affect the rate of flow of heat per unit area which would remain but would change total heat flux F_x to $u.v.\frac{m}{q} F_x$

APPENDIX 2

Scaling in a two dimensional heat conduction problem

The flow of heat by conduction in a two dimensional problem is governed by the following equations:-

$$\begin{cases} F_x = -K.\delta_y.\delta_z \frac{\partial \theta}{\partial x} & \ldots\ldots (5a) \\[2mm] F_y = -K.\delta x.\delta_z \frac{\partial \theta}{\partial y} & \ldots\ldots (5b) \end{cases}$$

and

$$\frac{\partial F_x}{\partial x}.\delta x + \frac{\partial F_y}{\partial y}.\delta y = -\rho S.\delta x.\delta_y.\delta_z \frac{\partial \theta}{\partial t} \qquad \ldots\ldots (6)$$

where the symbols are defined in Appendix 5.

If x and y be scaled by q, and K and the product ρS be scaled by m and n respectively, whilst θ be left unscaled, then the equations will remain consistent if modified as follows:-

$$m\,F_x = -\,m\,K.\delta_q y.\delta_z\,\frac{\partial\theta}{\partial_q x} \qquad \ldots\ldots\ (7a)$$

$$m.F_y = -\,m.K\,\delta_q x.\delta_z\,\frac{\partial\theta}{\partial_q y} \qquad \ldots\ldots(7b)$$

and

$$\frac{\partial\,m\,F_x}{\partial_q x}.\delta_q x + \frac{\partial\,m\,F_y}{\partial_q y}\,\delta_q y = -\,n.\rho S.\delta_q x\,\delta_q y\,\delta_z\,\frac{\partial\theta}{\frac{n q^2}{m}\partial t} \quad (8)$$

Thus time t becomes $\dfrac{n\,q^2}{m}\,t$ and rate of flow of heat per unit area

$$F_x\big/\delta_y.\delta_z \quad\text{or}\quad F_y\big/\delta_x.\delta_z \qquad \text{becomes} \quad \frac{m}{q}\left(\frac{F_x}{\delta_y.\delta_z}\right) \text{or } \frac{m}{q}\left(\frac{F_y}{\delta_x\delta_z}\right).$$

The z dimension does not enter into the thermal problem and may be scaled by any factor v. Such action would not affect the rate of flow of heat per unit area, which would remain $\dfrac{m}{q}\left(\dfrac{F_x}{\delta_y\delta_z}\right)$ or $\dfrac{m}{q}\left(\dfrac{F_y}{\delta_x\delta_z}\right)$

but it would change heat flux to vmF_x or vmF_y.

APPENDIX 3.

Scaling in a three dimensional heat conduction problem

The flow of heat by conduction in a three dimensional problem is governed by the following equations:-

$$F_x = -\,K\,\delta_y.\delta_z\,\frac{\partial\theta}{\partial x} \qquad \ldots\ldots\ (9a)$$

$$F_y = -K.\delta_x.\delta_z\,\frac{\partial\theta}{\partial y} \qquad \ldots\ldots(9b)$$

$$F_z = -K.\delta_x.\delta_y\,\frac{\partial\theta}{\partial z} \qquad \ldots\ldots(9c)$$

and $\dfrac{\partial F_x}{\partial x}\delta x + \dfrac{\partial F_y}{\partial y}\,\delta y + \dfrac{\partial F_z}{\partial z}\,\delta z = -\rho S.\delta x.\delta y.\delta z.\dfrac{\partial\theta}{\partial t}$ $\ldots\ldots(10)$

where the symbols are defined in Appendix 5.

If x, y and z be scaled by q, and K and the product ρS be scaled by m and n respectively, whilst Θ be left unscaled, then the equations will remain consistent if modified as follows:-

$$mq F_x = -mK\, \delta q y .\delta q z .\frac{\partial \theta}{\partial q x} \qquad \ldots\ldots \text{(11a)}$$

$$mq F_y = -mK\, \delta q x .\delta q z .\frac{\partial \theta}{\partial q y} \qquad \ldots\ldots \text{(11b)}$$

$$mq F_z = -mK\, \delta q x .\delta q y .\frac{\partial \theta}{\partial q z} \qquad \ldots\ldots \text{(11c)}$$

and

$$\frac{\partial mq F_x}{\partial q x}\, \delta q x + \frac{\partial mq F_y}{\partial q y}\, \delta q y + \frac{\partial m q F_z}{\partial q z}\, \delta q z = -n\, p s .\delta q x .\delta q y .\delta q z .\frac{\partial \theta}{\partial \frac{n q^2}{m} t} \qquad (12)$$

Thus time t becomes $\frac{n q^2}{m} t$ and rate of flow of heat per unit area

$$F_x / \delta_y .\delta_z \quad \text{or} \quad F_y / \delta x .\delta_z \quad \text{or} \quad F_z / \delta x .\delta_y$$

becomes $\frac{m}{q}\left(\frac{F_x}{\delta_y .\delta_z}\right)$ or $\frac{m}{q}\left(\frac{F_y}{\delta x .\delta_z}\right)$ or $\frac{m}{q}\left(\frac{F_z}{\delta x .\delta_y}\right)$

APPENDIX 4

Table of $a = b^{0.8}$

b	$a = b^{0.8}$	b	$a = b^{0.8}$	b	$a = b^{0.8}$
0.11	0.171	0.85	0.878	3.4	2.66
0.12	0.183	0.90	0.919	3.6	2.79
0.14	0.207	1.0	1.00	3.8	2.91
0.16	0.230	1.1	1.08	4.0	3.03
0.18	0.254	1.2	1.16	4.2	3.15
0.20	0.276	1.3	1.23	4.4	3.27
0.22	0.298	1.4	1.31	4.6	3.40
0.25	0.330	1.5	1.38	4.8	3.51
0.30	0.382	1.6	1.46	5.0	3.63
0.35	0.432	1.7	1.53	5.5	3.92
0.40	0.481	1.8	1.60	6.0	4.20
0.45	0.530	1.9	1.67	6.5	4.47
0.50	0.574	2.0	1.74	7.0	4.74
0.55	0.620	2.2	1.88	7.5	5.01
0.60	0.665	2.4	2.02	8.0	5.28
0.65	0.708	2.6	2.15	8.5	5.53
0.70	0.752	2.8	2.28	9.0	5.79
0.75	0.794	3.0	2.40	9.5	6.05
0.80	0.837	3.2	2.54	10.0	6.31

APPENDIX 5

List of symbols

x, y and z	Cartesian co-ordinates
t	Time
θ	Temperature rise
F_x, F_y and F_z	Components of heat flux, through defined areas, in the directions x, y and z
K	Thermal conductivity
ρ	Density
s	Specific heat
q	Scaling factor applied to dimensions in which there is a component of heat flow
u and v	Scaling factors applied to dimensions in which there is no component of heat flow
m	Scaling factor applied to thermal conductivity K
n	Scaling factor applied to product ρs